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A NOTE ON THE ABUNDANCE OF DIFFERENTIAL COMBINANTS IN A FUNDAMENTAL SYSTEM

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I. In a paper read before the section of algebra of the International Mathematical Congress at Toronto the present writer gave the definitions of the differential combinants of a set of p differential quantics,

$$f^{(r)} = a_{dx}^{(r)m} = (a_1^{(r)}\,dx_1 + \ldots + a_n^{(r)}\,dx_n)^m = (da^{(r)})^m, (r = 1, .., p), \quad (1)$$

$p \geqq n$, and developed general methods of generating these parameters. The special property of combinant differential parameters is their invariance when the forms themselves are subject to linear transformations with constant coefficients.[1] No theory of complete systems is yet known even in the binary case. In fact the problem of the fundamental system of algebraic[2] combinants is far from completion and one of the most recent writers on the latter[3] subject confined his discussion, in lieu of such theory, to statistical data, giving tables of important combinants of low orders for binary forms of various orders without emphasis upon finite systems. The significance of such data results from the importance of combinants in applications[4] of invariant theory to geometry.

The object of the present article is to indicate the methods which are available for the construction of corresponding tables of differential combinants. In particular it is shown that the number of combinant parameters in a complete system is necessarily large.

II. Suppose there are two binary ground-forms, f, g, as in (1),

$$f = (d\alpha)^m = \alpha_{dx}^m, \qquad g = (d\beta)^m = \beta_{dx}^m = \beta_{dx}'^m. \quad (2)$$

We construct a set, S, of algebraic combinants of f, g, a fundamental system if one is known for the given order m. These, and all combinants found, are complemented by multiplication by the proper powers of the resultant of f and g to make all invariants[5] absolute. We then apply to

S the following operations which generate combinant parameters from combinants:

(*i*) The jacobian of f and g or of any pair of combinants.

(*ii*) The total differential of any combinant.

(*iii*) The operation δ and its iterations:

$$\delta = \zeta^{(1)} \frac{\partial}{\partial x_1} + \zeta^{(2)} \frac{\partial}{\partial x_2}, \tag{3}$$

in which $\zeta^{(1)}$, $\zeta^{(2)}$ are any elements cogredient to dx_1, dx_2.

(*iv*) Polarization by

$$P = \zeta^{(1)} \frac{\partial}{\partial(dx_1)} + \zeta^{(2)} \frac{\partial}{\partial(dx_2)}. \tag{4}$$

Having generated all possible combinants of orders $\leqq t$, by (*i*), ..., (*iv*), we may form the algebraic simultaneous concomitant system of the totality and all parameters generated thus will be combinants of orders t or less.

The forms may be expressed either in Maschke's symbolic notation[5] or as actual functions.

III. Consider the problem of two quantics of the second order,

$$\begin{aligned} f &= (d\alpha)^2 = a_0(x_1, x_2)dx_1^2 + 2a_1(x_1, x_2)dx_1dx_2 + a_2(x_1, x_2)dx_2^2, \\ g &= (d\beta)^2 = b_0(x_1, x_2)dx_1^2 + 2b_1(x_1, x_2)dx_1dx_2 + b_2(x_1, x_2)dx_2^2. \end{aligned} \tag{5}$$

A complete system of algebraic combinants consists[6] of the first transvectant, J, of f and g and its discriminant D.

A combinant of order t of two m-ics is rational and integral in the determinants of order two of the matrix

$$\| a_{ix_1^{u-v}x_2^v}, b_{ix_1^{u-v}x_2^v} \|, (v = 0, \ldots, u; u = 0, \ldots, t; i = 0, \ldots, m), \tag{6}$$

where

$$a_{ix_1^{u-v}x_2^v} = \frac{\partial^u a_i}{\partial x_1^{u-v}\partial x_2^v}.$$

Each different set $(i, u-v, v)$ furnishes a row of the matrix. If we use the abbreviation

$$\begin{pmatrix} i & p & q \\ j & r & s \end{pmatrix} = a_{ix_1^p x_2^q} b_{jx_1^r x_2^s} - a_{jx_1^r x_2^s} b_{ix_1^p x_2^q}, \tag{7}$$

we have,

$$\begin{aligned} \frac{\partial}{\partial x_1}\begin{pmatrix} i & p & q \\ j & r & s \end{pmatrix} &= \begin{pmatrix} i & p+1 & q \\ j & r & s \end{pmatrix} + \begin{pmatrix} i & p & q \\ j & r+1 & s \end{pmatrix}, \\ \frac{\partial}{\partial x_2}\begin{pmatrix} i & p & q \\ j & r & s \end{pmatrix} &= \begin{pmatrix} i & p & q+1 \\ j & r & s \end{pmatrix} + \begin{pmatrix} i & p & q \\ j & r & s+1 \end{pmatrix}. \end{aligned} \tag{8}$$

The following is the complete list of combinants of orders $\leqq 1$, when[7]

$$\zeta^{(1)} = D^{-\frac{1}{4}}\frac{\partial u}{\partial x_2}, \quad \zeta^{(2)} = -D^{-\frac{1}{4}}\frac{\partial u}{\partial x_1}, \quad m = 2, \tag{9}$$

and u is an arbitrary function of x_1, x_2. Note that subscripts of symbols indicate partial derivatives. Thus,

$$\alpha_{1dx} = \frac{\partial^2\alpha}{\partial x_1^2}dx_1 + \frac{\partial^2\alpha}{\partial x_1\partial x_2}dx_2 = \alpha_{11}\,dx_1 + \alpha_{12}\,dx_2. \tag{10}$$

Symbolic forms

(1) $J = (\alpha\beta)\alpha_{dx}\beta_{dx}$, (2) $D = \frac{1}{2}(\alpha\beta)(\alpha'\beta')\,[(\alpha\beta')(\beta\alpha') + (\alpha\alpha')(\beta\beta')]$,

(3) $\frac{1}{4}(f, g) = (\alpha_{dx}\beta_{dx})\alpha_{dx}\beta_{dx}$, (4) $dD^{-\frac{1}{4}}J = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2\right)D^{-\frac{1}{4}}J$,

(5) $\delta D^{-\frac{1}{4}}J = (D^{-\frac{1}{4}}J, u)D^{-\frac{1}{4}}$

(6) Polars of the above forms by the operator P.

Leading coefficients

(1) $\binom{000}{100}$; (2) $\binom{000}{200}^2 - 4\binom{000}{100}\binom{100}{200}$; (3) $\binom{010}{001}$,

(4) $\left\{\binom{000}{100}\left[-\frac{1}{2}\binom{010}{200}\binom{000}{200} - \frac{1}{2}\binom{000}{210}\binom{000}{200} + \binom{010}{100}\binom{100}{200}\right.\right.$
$+ \binom{000}{110}\binom{100}{200} + \binom{000}{100}\binom{110}{200} + \binom{000}{100}\binom{100}{210}\Big] + D\Big[\binom{010}{100}$
$\left.\left. + \binom{000}{110}\right]\right\} D^{-\frac{5}{4}} = \zeta_1 D^{-\frac{5}{4}}$, (5) $(\zeta u)D^{-\frac{3}{2}}$; $\zeta_2 = \Big[-\frac{1}{2}\binom{000}{200}\binom{001}{200}$
$-\frac{1}{2}\binom{000}{200}\binom{000}{201} + \binom{001}{100}\binom{100}{200} + \binom{000}{101}\binom{100}{200} + \binom{000}{100}\binom{101}{200}$
$+ \binom{000}{100}\binom{100}{201}\Big]\binom{000}{100} + D\left[\binom{001}{100} + \binom{000}{101}\right]$.

IV. The paper mentioned in paragraph I is mostly theoretical, presenting the theory of differential combinants of general order of p forms in n variables. It deals, also, with a principle of translation analogous to that of Meyer[8] for rational curves. The generalization of (6), III, is fundamental: Combinants of order τ of the set (1) are functions of determinants of order p of the matrix

$$\left\| a_{i\ldots l}^{(1)} x_1^{s_1}\cdots x_n^{s_n}, \ \ldots\ldots\ldots, \ a_{i\ldots l}^{(p)} x_1^{s_1}\cdots x_n^{s_n} \right\|,$$

$(s_1 + \ldots + s_n = t;\ t = 0, \ldots, \tau;\ i + j + \ldots + l = m)$, in which the letters $a_{i\ldots l}^{(r)}$ are the actual coefficients of $f^{(r)}$.

[1] Sylvester, *Camb. and Dublin Math. J.*, **8**, 1853, 62–64.
[2] Gordan, *Math. Ann.*, **5**, 1872, 95–122.
[3] Shenton, *Amer. J. Math.*, **37**, 1915, 247–271.
[4] Rowe, *Trans. Am. Math. Soc.*, **12**, 1911, 295–310.
[5] Maschke, *Trans. Amer. Math. Soc.*, **4**, 1903, 446, 448.
[6] Glenn, *Theory of Invariants*, 1915, 167.
[7] *Encyc. der math. Wiss.*, **1** (IB2), 385.
[8] Meyer, *Apolarität und Rat. Curv.*, 1883, 18.

ON THE APPLICATION OF BOREL'S METHOD TO THE SUMMATION OF FOURIER'S SERIES[1]

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There are two ways of studying the relationship between various methods for the summation of divergent series. One consists in the attempt to determine directly whether or not one is more general than the other, and if this is not the case to determine under what conditions both methods apply and give the same sum to the series that is used. Another method involves the determination of the relative scope of the various processes in summing certain general types of series that are of fundamental importance in analysis. The former method is more exhaustive from the theoretical point of view; the latter is, perhaps, of greater practical interest.

The two most important types of series in analysis at the present time are power series and Fourier's series. It is well known that Cesàro's method will not sum a power series outside of its circle of convergence, whereas Borel's method applies everywhere within the polygon of summability. However, in the case where the circle of convergence is a natural boundary, Cesàro's method may be applicable at points on the circle of convergence where Borel's method fails. This phase of the relationship between the two methods may well be described by an illuminating remark made by G. H. Hardy[2] in another connection, namely, that "Borel's method, although more powerful than Cesàro's, is never more delicate, and often less so."

Cesàro's method has been found to be admirably adapted to the study of Fourier's series. It will give the proper sum for the Fourier's series of any continuous function at all points, and will sum the Fourier's series of any function having a Lebesgue integral to the value of the function, except perhaps at a set of points of measure zero. Since there is considerable similarity in the behavior of Fourier's series and the behavior of power series on the circle of convergence, it is natural to expect that Borel's

method will not be as effective as Cesàro's in dealing with the former type of series. That this is the case is proved by the results of the present paper.

The application of Borel's method to the summing of Fourier's series leads to the study of the behavior of the following integral[3]

$$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} [f(x+2t)+f(x-2t)]e^{-Xt^2}\frac{\sin Xt}{t}\,dt$$

as X becomes infinite. This integral is of the type termed by Lebesgue singular integrals. Its kernel

$$\varphi(X,t)=\frac{2}{\pi}e^{-Xt^2}\frac{\sin Xt}{t}$$

is such that

$$L_B(X)=\int_0^{\frac{\pi}{2}} |\varphi(X,t)|\,dt \qquad (1)$$

does not remain bounded as X becomes infinite. It follows therefore from a general theorem due to Lebesgue[4] that Borel's method will not sum the Fourier's series of every function having a Lebesgue integral or even of every continuous function.

Having found that Borel's method is less effective than Cesàro's in summing Fourier's series, we next wish to determine if it is more effective than ordinary convergence. The kernel of the singular integral that arises in the study of convergence, the well known Dirichlet's integral, is

$$\psi(n,t)=\frac{2}{\pi}\cdot\frac{\sin(2n+1)t}{\sin t}.$$

The values of

$$L(n)=\int_0^{\frac{\pi}{2}} |\psi(n,t)|\,dt \qquad (2)$$

for successive values of n are termed Lebesgue constants, this designation having been introduced by Fejér and adopted by later writers. The fact that they become infinite with n is tied up with the fact that the Fourier's series of a continuous function may diverge, and their order of infinity may be regarded as one form of measure of the degree of divergence that is possible in the case of the Fourier's series of a continuous function. The values for odd integral values of X of the function $L_B(X)$, defined by (1), may be regarded as the analogues of the Lebesgue constants for the application of Borel's method of summation. In spite of the factor e^{-Xt^2} in the integrand of (1), it may be shown that $L_B(n)$ is of the same order of infinity as $L(n)$, this order being that of $\log n$.

In view of this fact, one might not be surprised to find that Borel's method was no more effective in summing the Fourier's series of a continuous function than ordinary convergence. However, this is not the case, as can be shown by exhibiting an example of a continuous function whose Fourier's series is divergent but can nevertheless be summed by Borel's method. This example is obtained by slightly modifying an example given by Fejér[5] of a function whose Fourier's series diverges. Fejér's function is

$$\theta(x) = \sum_{n=1}^{\infty} \frac{\sin 2^{n^2} x}{n^2} \qquad (0 \leqq x \leqq \pi).$$

The cosine development of this function diverges for $x = 0$, and cannot be summed at this point by Borel's method. If, however, we set

$$\alpha(x, n) = 2^{\frac{1}{4}n^2} x \sin 2^{\frac{3}{4}n^2} \qquad (0 \leqq x \leqq 2^{-\frac{1}{4}n^2}),$$

$$\alpha(x, n) = \sin 2^{n^2} x \qquad (2^{-\frac{1}{4}n^2} < x \leqq \pi),$$

and form

$$F(x) = \sum_{n=1}^{\infty} \frac{\alpha(x, n)}{n^2} \qquad (0 \leqq x \leqq \pi),$$

we obtain a function whose cosine development diverges for $x = 0$ but can nevertheless be summed at that point by Borel's method.

The underlying reason for these facts may be briefly stated as follows. The essential cause for the divergence as well as the failure of Borel summability in the case of the Fourier's development of Fejér's function lies in the degree of rapidity with which the oscillations in the neighborhood of the origin of the individual terms of the series defining this function increase in frequency as we pass from one term to the next. The effect of the modifications we have made in the series defining Fejér's function is to remove such of the oscillations of each term as lie in a certain neighborhood of the origin, this neighborhood becoming steadily smaller as we pass to later terms. Thus we find in the resulting function oscillations of higher and higher frequency sufficiently near to the origin to cause divergence of the Fourier's development at that point. On the other hand we have removed enough of the oscillations in the immediate neighborhood of the origin to attain Borel summability of the development at that point, since the factor e^{-Xl^2} in the corresponding singular integral neutralizes the effect of the others.

It is apparent that the foregoing discussion, in addition to shedding further light on the relation between Borel and Cesàro summability, also furnishes one scheme for distinguishing between various types of continuous functions that have divergent Fourier's developments. It may be added that the scope of Euler's method[6] in the summation of Fourier's

series is the same as that of Borel's method, the resulting singular integrals being essentially equivalent. Hence the Fourier's development of the function we have defined above will also be summable by Euler's method at the point $x = 0$.

[1] Presented to the American Mathematical Society, December 29, 1924.

[2] Hardy, G. H., *Proc. London Math. Soc.*, Ser. 2, **11**, 1911 (1–16), p. 10.

[3] The integral first obtained is more complex but is essentially equivalent to the one given here.

[4] Lebesgue, H., *Toulouse Annales*, Ser. 3, **1**, 1910 (25–128), p. 70.

[5] Fejér, L., *Crelle's Journal*, **137**, 1910 (1–5).

[6] Cf. Knopp, K., *Math. Zeitschrift*, **15**, 1922 (226–253).

INTERSECTIONS OF COMPLEXES ON MANIFOLDS

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1. Let M_n, C_h, C_k be an orientable manifold and two complexes on it, the indices denoting the dimensions.[1] We propose to discuss what is meant in the most general case by the intersection of the complexes, and in particular by their Kronecker index when $h + k = n$.

We assume M_n covered with a C_n to be used as a basis for the definition of straightness and distances. If then C_h and C_k are made up exclusively with straight elements intersecting each other in the most general way possible, there is no essential difficulty. For straight cells in general relative position a procedure outlined elsewhere is applicable.[2] If the h and k cells of the complexes so behave, then the extension to them is also immediate, the sensed intersection being denoted by $C_h.C_k$, the Kronecker index when $k = n-h$, by $(C_h.C_k)$. Let Γ_{h-1}, Γ_{k-1} be the boundary cycles. We may then prove the basic Poincaré congruence

$$C_h.C_k \equiv (-1)^{n-k}\,\Gamma_{h-1}.C_k + C_h.\Gamma_{k-1}.$$

2. When the complexes are arbitrary our discussion will lead us only to a definition of a clear cut intersection if their boundaries do not intersect one another, and then $C_h.C_k$ is a cycle whose exact determination is obtained thus: We approximate C_h as closely as we please by a polyhedral complex C'_h such that there exist C_{h+1} and C°_h, respectively, very near C_h and its boundary, with a congruence

$$C_{h+1} \equiv C_h + C^\circ_h - C'_h. \qquad (1)$$

Moreover, C°_h contains the boundaries of both C_h and C'_h. This may

be done in such a way that: (*a*) The $(h-i)$-cells or C'_h are all on cells of no less than $n-i$ dimensions of C_n. (*b*) The vertices of C'_h may be slightly displaced without disturbing the situation. The same thing is done for C_k leading to a similar approximating C'_k possessing a well defined intersection with C'_h in the sense of No. 1, and a related C°_k. If the approximations are sufficient C°_h and C°_k will not intersect, neither will the boundaries of C'_h and C'_k, hence $C'_h.C'_k$ is a cycle.

3. I say that the mode of defining straightness being fixed $C'_h.C'_k$ is independent of the approximations provided they are sufficiently fine. Let first $\overline{C}'_h$ be another approximation analogous to C'_h, having also a well defined intersection with C'_k, and a corresponding congruence

$$\overline{C}_{h+1} \equiv C_h + \overline{C}^\circ_h - \overline{C}'_h.$$

Then

$$C_{h+1} - \overline{C}_{h+1} \equiv C'_h - \overline{C}'_h + C^\circ_h - C^\circ_h.$$

The left side may be approximated by a complex C'_{h+1} whose intersection with C'_k is well defined, having for boundary $C'_h - \overline{C}'_h$ and, in place of $C^\circ - \overline{C}^\circ_h$, a complex which does not meet C'_k. It follows

$$C'_{h+1} \equiv (C'_h - \overline{C}'_h).C'_k \sim 0, \text{ mod, } M_n.$$
$$\therefore C'_h.C'_k \sim C'_h.\overline{C}'_k.$$

If we now have to compare $C'_h.C'_k$ with $\overline{C}'_h.\overline{C}'_k$, intersection of two new polyhedral approximations, we draw C''_h, approximating C_h and having a well-defined intersection with both C'_k and $\overline{C}'_k$, then applying what precedes three times, we find

$$C'_h.C'_k \sim C''_h.C'_k \sim C''_h.\overline{C}'_k \sim \overline{C}'_h.\overline{C}'_k.$$
$$\therefore C'_h.C'_k \sim \overline{C}'_h.\overline{C}'_k.$$

For $h + k = n$ we have a uniquely defined Kronecker index $(C_h.C_{n-h})$.

The results of this section extend at once to intersections of more than two complexes. Suppose for examples that we deal with three and with obvious notations wish to show that

$$C'_h.C'_k.C'_l \sim \overline{C}'_h.\overline{C}'_k.\overline{C}'_l.$$

We introduce C''_h, approximation to C_h, in general position as to C'_k, C'_l, $\overline{C}'_k$, $\overline{C}'_l$. Then as above we show that

$$C'_h.C'_k.C'_l \sim C''_h.C'_k.C'_l; \quad C''_h.\overline{C}'_k.\overline{C}'_l \sim \overline{C}'_h.\overline{C}'_k.\overline{C}'_l.$$

Next we replace both C'_k and $\overline{C}'_k$ by C''_k in general position as to C''_h, C'_l, $\overline{C}'_l$ and find

$$C''_h.C''_k.C'_l \sim C''_h.C''_k.\overline{C}'_l$$

and so on.

4. I say finally that $C'_h.C'_k$ *is independent of the covering* C_n *and related definition of straightness.* This is the more difficult part of the discussion. We first modify C'_h and C'_k by adding tubes (boundaries of sets of polyhedral cells in linear order) so as to obtain complexes whose sets of ordinary points are connected. This may be done without affecting $C'_h.C'_k$. A similar operation on C'_k alone will turn $C'_h.C'_k$ into a cycle having the same property. It may then be shown with ease that all $h + k - n$ cycles on $C'_h.C'_k$ or on a suitably defined neighborhood complex N of the cycle are $\sim$ multiples of $C'_h.C'_k$ itself. When $h + k = n$ this will allow us to reduce actual intersections to a single point P. We may in fact arrange matters so that P is an arbitrarily assigned point and that the intersecting cells through P are also assigned.

By means of this we show first that when the covering complex is changed from C_n to $\bar{C}_n$, and the definition of straightness is also altered, then the Kronecker index $(C'_h.C'_{n-h})$ may only be multiplied by ± 1, the sign depending upon h alone. Then using certain auxiliary complexes of the neighborhood of P we may show that this sign is equally independent of h, hence with suitable conventions may be taken as $+$. The Kronecker index is then unchanged—its value it is that we denote by $(C_h.C_{n-h})$. This may be extended to the index of intersection of several manifolds.

Let now $h + k > n$ and let the approximations by means of $\bar{C}_n$ be $\bar{C}'_h$, $\bar{C}'_k$. In deriving them we may clearly replace C_h, C_k by C'_h, C'_k. We carry this approximation so far that $\bar{C}'_h.\bar{C}'_k$ lies on N. Hence

$$\bar{C}'_h.\bar{C}'_k \sim t.C'_h.C'_k.$$

Applying then the result already obtained for the index to the intersection with a suitably chosen cell that meets $C'_h.C'_k$ at a unique point, and whose construction is easy, we obtain $t = 1$. This proves that to C_h and C_k may be attached a uniquely defined cycle which we denote then by $C_h.C_k$. The extension to several complexes is immediate. These intersection-cycles may be combined with new complexes so as to yield new cycles, eventually Kronecker indices according to rules which may be formulated with ease.

[1] The notations, terminology and the like are essentially as in Veblen's *Colloquium Lectures on Analysis Situs.*

[2] L'Analysis Situs et la Géométrie algébrique, p. 10. Monograph of the Borel Series, Paris, Gauthiers-Villars (1924).

CONTINUOUS TRANSFORMATIONS OF MANIFOLDS

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1. I propose to communicate here further results along the line of a previous Note.[1]

The product of two complexes C_p, C_q, in the sense of Steinitz (image of their point-pairs) will be written $C_p \times C_q$, and is not to be confused with their intersection $C_p.C_q$.

Let W_n, W'_n be the manifolds of the first Note, W'_n a copy of W. To assign a transformation T of W_n into itself is equivalent to assigning a correspondence between point-pairs A, B of W_n and W'_n, where B is the image of $T.A$ on W'_n. The complete representation of T is then given by the point set $A \times B$ of $W_n \times W'_n$. We impose upon T the unique restriction that the set shall constitute an n-cycle Γ_n of the product manifold—practically a mild restriction indeed.

Let Γ'_n correspond to T'. We are after the numbers $(\Gamma_n.\Gamma'_n)$ for whose interpretation see loc. cit.

2. Denote generically by C_h, C'_h complexes, respectively, in W_n, W'_n. Then by means of indicatrices and in the light of another Note[2]

$$(C_h \times C'_k.C_{n-h} \times C'_{n-k}) = (-1)^{k(n-h)} \quad (C_h.C_{n-h})\,(C'_k.C'_{n-k})$$

$$(W_n \times B.A \times W'_n) = +1$$

$$(C_h \times C'_k \,.\, C_{n-h-i} \times C'_{n-k+i}) = 0.$$

3. Let R_μ be the μ-th connectivity index of W_n ($1 + R_\mu$ is the μ-th Betti number), and let $\gamma_\mu^1, \gamma_\mu^2, \ldots, \gamma_\mu^{R_\mu}$, be the cycles of a canonical fundamental set for the μ-th dimensionality, which are independent.[3] Let also δ_μ^i be the image of γ_μ^i on W'_n. We shall denote homologies "with division allowed" (i.e., with zero divisors neglected) by $\approx$ in place of the usual $\sim$. It may then be shown that the cycles $\gamma_\mu^i \times \delta_{n-\mu}^j$ constitute a fundamental set for the relation $\approx$ and the n cycles of $W_n \times W'_n$. Hence

$$\Gamma_n \approx \sum \epsilon_\mu^{ij}\, \gamma_\mu^i \times \delta_{n-\mu}^j, \quad \text{mod } W_n \times W'_n. \tag{1}$$

There is a similar relation for Γ'_n with the ϵ's replaced by other integers say η_μ^{ij}.

4. By combining the formulas of No. 2, with an important theorem of Veblen's on the Kronecker indices of fundamental sets,[4] we may derive, readily enough, a formula for $(\Gamma_n \,.\, \Gamma'_n)$ in terms of the ϵ's and η's. It is, however, of distinctly more interest to obtain the expression of the index in terms of the coefficients in the transformations induced by T and T' on the cycles of W_n. Thus for T:

$$\bar{\gamma}_\mu^i \approx \sum_j \alpha_\mu^{ij} \gamma_\mu^j, \quad \text{mod } W_n,$$

would correspond to the effect on the μ-cycles. There will be similar formulas for T' with β's in place of α's.

If $\bar{\gamma}_\mu \approx T \cdot \gamma_\mu$, we prove by means of indicatrices this all important relation:

$$(\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu}) = (-1)^\mu (\gamma_\mu \cdot \gamma_{n-\mu}), \tag{2}$$

the indices being computed as to $W_n \times W_n'$ and W_n, respectively. If we substitute for Γ_n its expression from No. 3, for γ_μ and $\delta_{n-\mu}$ the cycles of the canonical sets and apply Veblen's theorem we obtain the desired expression of the ϵ's in terms of the α's, hence also of the η's in terms of the β's; whence ultimately:

(*a*) $n \not\equiv 2$, mod 4

$$(\Gamma_n \cdot \Gamma_n') = \sum_{\mu=0}^{n} (-1)^\mu \sum_{i,j=1}^{R_\mu} \alpha_\mu^{ij} \beta_{n-\mu}^{ij}. \tag{3}$$

(*b*) $n = 4m + 2$

$$\begin{aligned}(\Gamma_n \cdot \Gamma_n') = &\sum_{\mu=0}^{2m} (-1)^\mu \sum_{i,j=1}^{R_\mu} (\alpha_\mu^{ij} \beta_{n-\mu}^{ij} + \ldots) \\ &+ \sum_{h,k=1}^{\frac{1}{2}R_{2m+1}} (\alpha_{2m+1}^{2h-1,\,2k} \beta_{2m+1}^{2h,\,2k-1} - \alpha_{2m+1}^{2h-1,\,2k-1} \beta_{2m+1}^{2h,\,2k} + \ldots),\end{aligned} \tag{4}$$

the terms omitted being such as to make the formula symmetrical.

5. *Interesting special cases.*—(*a*) *Fixed points of a single* T. It will correspond to $T' = 1$, $\beta_\mu^{ij} = 0$ of $i \neq j$, $\beta_\mu^{ij} = 1$. Hence if Γ_n° corresponds to the identity:

$$(\Gamma_n^\circ \cdot \Gamma_n) = \sum_{\mu=0}^{n} (-1)^\mu \sum_{i=1}^{R_\mu} \alpha_\mu^{ii},$$

which, as it happens, holds whether the fundamental sets are canonical or not. This includes for example Alexander's recent results for $n = 2$.[5]

(*b*) *T' is of the class of T and n is odd.* Then $(\Gamma_n \cdot \Gamma_n') = 0$.

(*c*) *T is of the class of a deformation.* Then

$$(\Gamma_n^\circ \cdot \Gamma_n) = \sum (-1)^\mu R_\mu.$$

Hence this striking theorem: *The minimum number of fixed points under a T of the class of the identity is equal in absolute value to the Euler characteristic.*

In point of fact this remains true if T operating on any cycle, merely increases it by a zero divisor. Again when n is odd, the number is zero. For $n = 2$ and if p is the genus of the W_2, the number is $2p-2$ as found by Birkhoff.[6]

(*d*) *Hurwitz's coincidence formula.* Thanks to a theorem proved elsewhere[7] it is a direct corollary of (4) in the case $n = 2$.[8]

In conclusion let us state that most results obtained along this line by Brouwer and his school follow from (3) and (4).

[1] These PROCEEDINGS, **9**, p. 99 (1923).

[2] In the present volume of these PROCEEDINGS.

[3] See my Borel Series Monograph: L'Analysis Situs et la géométrie algébrique, p. 15 (1924).

[4] *Trans. Amer. Math. Soc.*, **25** (1923)540.

[5] *Trans. Amer. Math. Soc.*, **25** (1923)173.

[6] *Trans. Amer. Math. Soc.*, **18** (1917)287.

[7] *Monograph*, p. 19.

[8] The product manifold was indeed used for the first time in the same manner as here in connection with algebraic correspondences by Severi in his well-known paper of the *Torino Memorie* of 1903.

LAWS OF RECIPROCITY AND THE FIRST CASE OF FERMAT'S LAST THEOREM

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Suppose that

$$x^p + y^p + z^p = 0 \qquad (1)$$

is satisfied in rational integers x, y and z, none zero and each prime to the odd prime p. Then

$$(x + \alpha y)(x + \alpha^2 y) \ldots (x + \alpha^{p-1} y) = u^p$$

where $\alpha = e^{2i\pi/p}$, and u is a rational integer. It follows from the unique decomposition of an ideal into its prime ideal factors that the ideal $(x + \alpha^a y)$ is the p^{th} power of an ideal in the field $\Omega(\alpha)$, $a \not\equiv 0 \pmod{p}$. If ω is an integer in $\Omega(\alpha)$, $\mathfrak{q}$ an ideal prime in $\Omega(\alpha)$ which is prime to (ω) and (p) then we set

$$\omega^{\frac{N(\mathfrak{q})-1}{p}} \equiv \left\{\frac{\omega}{\mathfrak{q}}\right\} \pmod{\mathfrak{q}},$$

where $N(\mathfrak{q})$ is the norm of $\mathfrak{q}$ and $\{\omega/\mathfrak{q}\}$ is some power of α. It follows that ω is congruent to a p^{th} power in Ω, modulo $\mathfrak{q}$, if and only if,

$$\left\{\frac{\omega}{\mathfrak{q}}\right\} = 1.$$

Also if $Q = \mathfrak{q}_1 \mathfrak{q}_2 \ldots \mathfrak{q}_s$, where the $\mathfrak{q}$'s are prime ideals prime to (ω) and (p), then

$$\left\{\frac{\omega}{Q}\right\} = \left\{\frac{\omega}{\mathfrak{q}_1}\right\}\left\{\frac{\omega}{\mathfrak{q}_2}\right\} \cdots \left\{\frac{\omega}{\mathfrak{q}_s}\right\}.$$

Consequently since $(x + \alpha^a y)$ is the p^{th} power of an ideal in $\Omega(\alpha)$ then

$$\{\omega/(x + \alpha^a y)\} = 1.$$

Hilbert[1] defined the norm residue symbol

$$\left\{\frac{\nu, \mu}{\mathfrak{p}}\right\} = \alpha^L \tag{2}$$

where

$$L \equiv \sum_{s=1}^{p-1} (-1)^{s-1} l^{(s)}(\nu)\, l^{(p-s)}(\mu) \pmod{p},$$

$$l^{(g)}(\omega) = \left[\frac{d^g \log(\omega(e^v))}{dv^g}\right]_{v=0}$$

$g = 1, 2, \ldots, p-1$; $\mathfrak{p} = (1-\alpha)$; $\nu \equiv 1 \pmod{\mathfrak{p}}$, $\mu \equiv 1 \pmod{\mathfrak{p}}$; ν and μ integers in the field $\Omega(\alpha)$, $\alpha = e^{2i\pi/p}$.

The symbol $\omega(x)$ is defined follows: If

$$\omega = c_0 + c_1 \alpha + \ldots + c_{p-2} \alpha^{p-2},$$

then

$$\omega(x) = \sum_{r=0}^{p-2} c_r x^r - \frac{\sum_{r=0}^{p-2} c_r - 1}{p} \cdot \frac{x^p - 1}{x - 1}. \tag{3}$$

If ν and μ are any integers in the field prime to $\mathfrak{p}$ then

$$\left\{\frac{\nu, \mu}{\mathfrak{p}}\right\} = \left\{\frac{\nu, \mu^{p-1}}{\mathfrak{p}}\right\}. \tag{4}$$

Now if (γ) is a principal ideal and the p^{th} power of any ideal in $\Omega(\alpha)$ then (γ), $(x + \alpha^a y)$ being prime to each other and to $\mathfrak{p}$,

$$\left\{\frac{\gamma}{x + \alpha^a y}\right\} = 1 = \left\{\frac{x + \alpha^a y}{\gamma}\right\}. \tag{5}$$

Furtwängler[2] gave a law of reciprocity between two integers in a field containing $\Omega(\alpha)$, which for the special case where the field is $\Omega(\alpha)$ itself becomes

$$\left\{\frac{\omega}{\theta}\right\}\left\{\frac{\theta}{\omega}\right\}^{-1} = \left\{\frac{\theta, \omega}{\mathfrak{p}}\right\}, \tag{6}$$

ω and θ being any two integers in $\Omega(\alpha)$ such that (ω) and (θ) are prime to each other and to $\mathfrak{p}$. Setting $\omega = \gamma^{p-1}$ and $\theta = (x + \alpha^a y)^{p-1}$ then (4), (5) and (6) give

$$\left\{\frac{x + \alpha^a y, \gamma}{\mathfrak{p}}\right\} = 1$$

or

$$L \equiv 0 \pmod{p}, \tag{7}$$

for

$$\gamma = \theta, \mu = \omega.$$

Let

$$\omega = \gamma^{p-1} = \sum_{r=0}^{p-2} c_r \alpha^r, (x + \alpha^a y)^{p-1} = \sum_{r=0}^{p-2} d_r \alpha^r = \theta.$$

Then (Hilbert,[1] p. 413)

$$l^{(g)}(\omega) \equiv \left[\frac{d^g (\log \sum_{r=0}^{p-2} c_r e^{vr})}{dv^g}\right]_{v=0} \pmod{p},$$

$g = 1, 2, \ldots, p-2$, and similar relations for θ. Also if we set $\gamma(e^v)$ for the function obtained when α is replaced by e^v in γ then evidently

$$\left[\frac{d^g \log \sum_{r=0}^{p-2} c_r e^{vr}}{dv^g}\right]_{v=0} \equiv -\left[\frac{d^g \log \gamma(e^v)}{dv^g}\right]_{v=0} \pmod{p}, \tag{8}$$

$$\begin{aligned}\left[\frac{d^g \log \sum_{r=0}^{p-2} d_r e^{vr}}{dv^g}\right]_{v=0} &\equiv -\left[\frac{d^g \log (x + e^{av} y)}{dv^g}\right]_{v=0} \pmod{p}, \\ &\equiv -a^g \left[\frac{d^g \log (x + e^v y)}{dv^g}\right]_{v=0} \pmod{p}.\end{aligned} \tag{9}$$

Now also

$$\left[\frac{d^{p-1} \log \gamma^{p-1}(e^v)}{dv^{p-1}}\right]_{v=0} \equiv l^{(p-1)}(\omega) + \frac{1 - \gamma^{p-1}(1)}{p}, \tag{10}$$

modulo p. We also have

$$N(\gamma^{p-1}) = b^{p(p-1)}$$

where b is an integer, since (γ) is the p^{th} power of an ideal in $\Omega(\alpha)$. Hence if u is an indeterminate

$$[\gamma(u)\gamma(u^2) \ldots \gamma(u^{p-1})]^{p-1} = b^{p(p-1)} + k \cdot \frac{u^p - 1}{u - 1} + V(u)(u^p - 1), \tag{11}$$

where k is a rational integer, and $V(u)$ is a polynomial in u with rational integral coefficients. Set $u = 1$, then

$$(\gamma(1))^{(p-1)^2} \equiv 1 + kp \pmod{p^2}$$

since $b^{p(p-1)} \equiv 1 \pmod{p^2}$, whence

$$\frac{(\gamma(1))^{(p-1)^2} - 1}{p} \equiv k \pmod{p}.$$

Now in (11) set also $u = e^v$, take logarithms and differentiate $(p-1)$ times with respect to v, then

$$(1^{p-1} + 2^{p-1} + \ldots + (p-1)^{p-1}) \left[\frac{d^{p-1}(\log \gamma (e^v)^{p-1})}{dv^{p-1}}\right]_{v=0} \equiv k(1^{p-1} + 2^{p-1} + \ldots + (p-1)^{p-1}) \pmod{p},$$

or

$$\left[\frac{d^{p-1}(\log \gamma (e^v)^{p-1})}{dv^{p-1}}\right]_{v=0} \equiv \frac{(\gamma(1))^{(p-1)^2} - 1}{p} \pmod{p}. \quad (12)$$

Now if

$$(\gamma(1))^{p-1} = 1 + rp$$

then

$$(\gamma(1))^{(p-1)^2} \equiv 1 + r(p-1)p \pmod{p^2}$$

and

$$\frac{\gamma(1)^{(p-1)^2} - 1}{p} \equiv -\frac{(\gamma(1))^{p-1} - 1}{p} \pmod{p},$$

and from (10) with (12),

$$l^{(p-1)}(\omega) \equiv 0 \pmod{p}. \quad (13)$$

We may prove in a similar manner that

$$l^{p-1}(\theta) \equiv 0 \pmod{p}. \quad (14)$$

Hence (7) gives

$$\sum_{s=2}^{p-2} (-1)^{s-1} l^{(s)}(\omega)\, l^{(p-s)}(\theta) \equiv 0 \pmod{p},$$

or from (8) and (9)

$$\sum_{s=2}^{p-2} (-1)^{s-1} a^s \left[\frac{d^s \log(x + e^v y)}{dv^s}\right]_{v=0} \left[\frac{d^{p-s} \log \gamma (e^v)}{dv^{p-s}}\right]_{v=0} \equiv 0 \pmod{p}.$$

Set $a = 1, 2, 3, \ldots, p-2$ in turn, we obtain $(p-2)$ congruences, and since the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 2 & 2^2 & 2^3 & \ldots & 2^{p-2} \\ & & \cdot & & \\ (p-2) & (p-2)^2 & (p-2)^3 & \ldots & (p-2)^{p-2} \end{vmatrix} = \prod_{i,j} (i-j)\ (p-2)!$$

$$i, j, = 1, 2, \ldots, p-2, \quad i>j$$

is not divisible by p, then we have

$$\left[\frac{d^s \log (x + e^v y)}{dv^s}\right]_{v=0} \left[\frac{d^{p-s} \log \gamma (e^v)}{dv^{p-s}}\right]_{v=0} \equiv 0 \pmod{p},$$

$s = 2, 3, \ldots, p-2$. But we also have (Vandiver[3]), if we set $-x/y = t$,

$$f_s(t) \left[\frac{d^{p-s} \log \gamma (e^v)}{dv^{p-s}}\right]_{v=0} \equiv 0 \pmod{p}, \qquad (15)$$

$s = 2, 3, \ldots, p-2$; $f_s(t) = \sum_{i=1}^{p-1} i^{s-1} t^i$. We may then state the

THEOREM. *If* (1) *is satisfied in rational integers* x, y, z, *none zero and all prime to the odd prime* p *then*

$$f_{p-n}(t) \left[\frac{d^n \log \gamma(e^v)}{dv^n}\right]_{v=0} \equiv 0 \pmod{p},$$

$n = 2, 3, \ldots, p-2$; $-t = x/y$. *Here* $\gamma(\alpha)$ *is an integer in the field* $\Omega(\alpha)$ *and the principal ideal* $(\gamma(\alpha))$ *is the* p^{th} *power of any ideal in* $\Omega(\alpha)$ *which is prime to* (z) *and* (p).

If there exists two solutions of (1) say

$$x^p + y^p + z^p = 0$$
$$x_1^p + y_1^p + z_1^p = 0$$

with $xyz \not\equiv 0 \pmod{p}$, $x_1 y_1 z_1 \not\equiv 0 \pmod{p}$, where z is prime to z_1, then in the above theorem we may take $\gamma(\alpha) = x_1 + \alpha y_1$ and we obtain

COROLLARY I. *If there exists in addition to the solution* (x, y, z) *of* (1) $(xyz$ *prime to* $p)$ *another solution* (x_1, y_1, z_1) z_1 *prime to* z *and* $x_1 y_1 z_1 \not\equiv 0$ *(mod* p*), then*

$$f_{p-n}(t) f_n(t_1) \equiv 0 \pmod{p}$$

$t_1 = -x_1/y_1$, $n = 2, 3, \ldots, p-2$.

If we take $\gamma(\alpha) = z + \alpha y$, which is prime to $(x + \alpha y)$ and to (p), then we have

COROLLARY II. *If* (1) *is satisfied for* $xyz \not\equiv 0$ *(mod* p*)*

then

$$f_{p-n}(t)\, f_n(1-t) \equiv 0 \pmod{p}$$

$n = 2, 3, \ldots, p-2$.

Put $x + \alpha^{p-1}y = \gamma(\alpha)$, this is prime to $x + \alpha^a y$, $0 < a < p-1$, so that we obtain

COROLLARY III. *If* (1) *is satisfied for* $xyz \not\equiv 0 \pmod{p}$, then

$$f_n(t)\, f_{p-n}(t) \equiv 0 \pmod{p}, \tag{16}$$

$n = 2, 3, \ldots, p-2$.

Cf. Mirimanoff.[4]

We have

$$\left[\frac{d^2 \log (x + e^v y)}{dv^2}\right]_{v=0} = \frac{xy}{(x+y)^2}.$$

Hence the theorem gives

$$\left[\frac{d^{p-2} \log \lambda (e^v)}{dv^{p-2}}\right]_{v=0} \equiv 0 \pmod{p}$$

for all ideals $(\lambda(\alpha))$ which are p^{th} powers of ideals in the field prime to (z) and (p). Using $x + \alpha z$ in lieu of $x + \alpha y$ we obtain the same result for ideals which are not prime to (z). We deal in a similar manner with the case $n = 3$ in the theorem, and find results which are included in the following:

COROLLARY IV. *If* (1) *is satisfied in integers prime to the odd prime* p *then*

$$\left[\frac{d^{p-a} \log \lambda(e^v)}{dv^{p-a}}\right]_{v=0} \equiv 0 \pmod{p},$$

$a = 2$ or 3, and $(\lambda(\alpha))$ is the p^{th} power of any ideal in the field $\Omega(\alpha)$ which is prime to p.

Note that the latter criteria are independent of x, y and z. From (1) we have also

$$f_{p-1}(t) \equiv 0 \pmod{p} \tag{17}$$

for $t = -x/y$. This well known criterion may be deduced by elementary means. Mirimanoff deduced from the Kummer criteria

$$B_a f_{p-2a}(t) \equiv 0 \pmod{p}$$

$a = 1, 2, \ldots, p-3/2$, and where $B_1 = 1/6$, $B_2 = 1/30$, ..., etc., are the numbers of Bernoulli, the criteria (16) using also (17). Conversely we may obtain from his relations a new derivation of the Kummer criteria by the use of (16) and (17).

The Theorem and first three corollaries of this paper were previously stated without proof by the writer.[5]

All the results given here have been known to the writer since 1917, but he was enabled to put the article in final form through a grant, relieving him of the duties of teaching, from the Heckscher Foundation for the Advancement of Research, established by August Heckscher at Cornell University.

[1] Bericht über die Theorie der algebraischen Zahlkorper, *Jahresbericht D.M.V.*, 1894, 413.

[2] *Math. Ann.*, **72**, 1909, 386.

[3] *Ann. Math.*, **21**, 1919, 76.

[4] *J. Mathematik, Crelle*, **128**, 1905, 45–68.

[5] *Amer. Math. Soc. Bull.*, **28**, 1922, 259–60.

A SPECIES OF MUSA IN THE TERTIARY OF SOUTH AMERICA

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The original home of the banana (Musa) has always been a disputed problem, most authors inclining to the belief that it was in the southeastern Asiatic region, since there are so many cultivated varieties as well as wild forms in that region at the present time, and since the equivalent word occurs in Sanscrit, Arabic, etc.

More particularly, botanists have, almost without exception, disputed that the genus Musa was indigenous in the Western Hemisphere, or was cultivated here prior to its introduction into Hispaniola from the Canary Islands in 1516, as related by Oviedo in 1556.

Alexander von Humboldt was the first to question this belief and to maintain that the banana was probably a native of America in pre-Spanish times, and in recent years O. F. Cook has argued that it was an aboriginal American crop-plant, originally cultivated as a root crop, the valuable pulpy fruits normally seedless being a result of hybridization, since the so-called wild seed-bearing species have non-edible fruits more or less similar to the capsular fruits of all the other genera of the family.

I have received the fossil seeds of a species of Musa from Dr. Maurice A. Rollot, who collected them from the coal measures of the Cerros de Guadalupe and Montserrate, which form a part of the eastern upland border of the Sabana of Bogota in Colombia. The altitude is about 9000 feet and the geological horizon is probably Oligocene, although there is some doubt as to its precise age.

These more or less compressed and highly lignified fossil seeds are larger than those of any of the existing Malayasian species known to me, but may be matched by the seeds of several existing African species, among which they most resemble those of the so-called Abyssinian banana, *Musa ensete* Gmelin.

Banana-like leaves, described by paleobotanists under the name of Musaphyllum, have long been known from the Eocene of the western United States and from the later Tertiary of Central and South America, and Europe, but it is altogether impossible to distinguish the leaves of Musa from those of the related American genus Heliconia. In the case of the seeds, however, this uncertainty disappears, for the seeds of Musa are perfectly characteristic and strikingly different from those of all the other genera of the family Musaceae.

The present discovery shows that Musa was an undoubted member of American Tertiary floras, and although it does not necessarily prove that the banana was brought into cultivation by the aborigines in the Western Hemisphere, it lends probability to such a belief, and in a measure serves to substantiate the statement of the Peruvian author Garcillaso de la Vega (1530–1568) that the banana was one of the staples of the aborigines before the discovery of America, or that of Montesinos (1527), as quoted by the historian Prescott, that the natives of Tumbez brought bananas as an offering to Pizarro when he disembarked.

OUTLINES OF THE HISTORY OF THE RECENT FAUNA OF PALAEARCTIC ASIA

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I wish to thank the Local Committee of the Academy for the opportunity of presenting to you my views concerning the history of the recent fauna of Siberia and High Asia. These views are the result of studies based in part on my field work in the Kirghiz Steppe, in southern Siberia and in N. W. Mongolia.

At present, most zoögeographers adopt the divisions of Palaearctic Asia into two subregions: the Northern, which extends across the Eurasian continent from the Pacific to the Atlantic, and the more southern, or High Asian. For the Northern subregion, most of the Russian zoögeographers adopt a further subdivision, proposed by N. Severtzoff, into three provinces: one, the Tundra or northern barren grounds; two, the Taïga or coniferous woodland; and, three, the Steppe and Forest-Steppe—simple

prairie or prairie with scattered forests. These divisions run across the continent along the parallels of latitude, but the Steppe province is largely interrupted in southern Siberia. Of the divisions of the High Asian subregion, I shall mention here only the Mongolo-Tibetan province, which embraces practically the whole elevated desert region of Central Asia.

These divisions are based on the recent distribution of the fauna without any relation to its history, and the principal divisions are considered those which are characterized simply by a large number of species. They have been elaborated primarily for birds, but hold good for other groups also.

Historical and causal understanding of these relations may be arrived at by grouping together the species having the same distribution, and by taking into consideration their ecology, the amount of their structural differences, and the facts of palaeogeography. Starting with the study of birds, I have found that nearly one half of the fauna of every locality in Palaearctic Asia consists of species which may be termed transpalaearctic, since they extend from ocean to ocean: some of them are also widely distributed from north to south, whereas others are restricted to one of the zonal provinces. Such distributions are of recent date, being in harmony with the landscape and other conditions of the present time. The remainder consists of groups of more restricted distribution. For Siberia, leaving out the transpalaearctic elements, the important groups are: on the east, the Transyenissean, or East-Siberian, group with its subdivisions, and on the west, the West-Palaearctic and West-Siberian groups. The demarcation line between them runs, roughly speaking, along the Yenissei River and then sweeps westward round the Altai highland. I have found that the Transyenissean species are strongly characterized, some presenting characters of relatively high antiquity; moreover, this group is intimately connected with the fauna of High Asia, many species reaching southward as far as the southern limits of the palaearctic; some of the Transyenissean forms are surely of southern origin. On the other hand, the fauna of western Siberia is in the main common with that of Europe; its distinctive forms are not numerous and are feebly characterized. I found this true for birds and butterflies; by others, the same features of East Siberian and West Siberian faunae have been pointed out for dragonflies, Trichoptera, and mammals.

These facts find no explanation in recent conditions, but they are in full harmony with the palaeogeography of these regions. The transyenissean elements, strongly characterized, with features of antiquity and presenting relations to the fauna of High Asia—in spite of present differences of physical geography—correspond to the territory of that part of the Eurasian dry land which is known as the Angara continent. Western Siberia is a new dry land which emerged only at the beginning of the second half

of the Tertiary, and underwent considerable restriction of its area in later times. With this corresponds the paucity and prevailingly youthful character of its endemic forms. The transpalaearctic distributions which prevail now are features of a later time, and for some of the animals involved traces of their Angaran, or western, origin may be found.

Details of present distribution, together with facts of geology, soil science, botany and palaeontology show the following course in the later history of the country. The pliocene climate was humid and mild; fall of temperature resulted in local glaciations and has conditioned an exchange of the faunas between the arctic zone and South-Siberian mountains. The glacial period was followed by a dry period, when the conditions of a dry climate and the elements of the steppe reached northward even beyond 60° n. lat., and at that time the saiga antelope roamed as far north as the Vilui River; about the same time, a northward migration of the alpine elements of High Asia took place and the wild sheep of Altai extended nearly 400 miles further north than now. After that came a spreading of the woodland, which has now invaded the alpine zone and the steppe landscape, and has broken up the province of the steppe. The climate did not, however, return to the favorable type of the preglacial time, being generally cold and dry. Owing to these factors, many gaps in distribution peculiar to Siberia have been caused. This type of climate, with normally hot summer but exceedingly cold, though snowless, winter, is due to the peculiarities of the distribution of atmospheric pressures. And this, in turn, is conditioned by the proximity of the extensive elevated plateaus of High Asia. Their upheaval became fatal for the climate of Siberia.

The fauna of High Asia is also strongly characterized, harmonizing well with the old age of this dry land. Prevalent in the fauna are biological types characteristic of desert, of mountain, and especially of dry highland. Sharp characteristics, and in some cases geographical distribution, of these biological types tell of a long duration of recent conditions. But at the same time, the distribution of the fauna presents many gaps and many isolated remnants, these isolated colonies often having failed, as yet, to develop local characters. And all species which present an interrupted distribution are types characteristic of a more fertile landscape. These facts tell clearly of the extensive devastations to which the fauna has been recently subjected.

My interpretation is, that present ecological types, and corresponding environment, are of an old origin. But that their present dominance is a new fact. I deny completely the theory of a large inland basin within the territory of High Asia in the Tertiary, and I consider the country as having presented the conditions of a continental climate, the land being more fertile, though with patches of desert here and there, and the rainfall more abundant than now.

Distribution of the true pheasants permits the location of one of these primary isolated foci of the desert, namely—in the western Gobi, east of the lake Lobnor. Extension of the desert began in Pliocene (though the eastern Gobi has not yet been invaded), whereas N. W. Mongolia and Dzungaria have been conquered by the desert only much later—N. W. Mongolia about the close of glacial time. In its recent fauna, local desert types are just beginning to develop, strongly characterized desert types are all immigrants, and well characterized local forms are not desert types.

Thus, speaking briefly, we have for Siberia elements of old Angaran fauna overlaid by modernized transpalaearctic distributions, the last influenced, in turn, by the glaciation, by the transient spreading of the faunas of the Steppe and High Asia, and by the general deterioration of the climate, which never returned to its preglacial type. For central Asia, we have encroachment of desert and also upheavals which resulted in a spreading of the types now prevailing, and in the suppression of others. The Mongolo-Tibetan province in its present zoölogical character and limits is a new phenomenon, based on hypertrophy of some of its ancient elements.

Concerning the southeast corner of palaearctic Asia, I agree with N. Severtzoff that it presents a remnant of the old palaearctic fauna. At present, being isolated by enormous upheavals, it exists under conditions which favor the invasion into it of the Indo-Malayan fauna. Many of the palaearctic forms and some even of the more northern species seem to have originated here.

I have no time to consider in detail the complicated faunal relations between Siberia and North America. It is, however, to be noted that the interchange was very prolonged, some of the forms common to both continents (or represented by related types) having originated in Asia, others in America, and some almost certainly within the territory of the former land connection itself—a land which may be called Beltringia. One of the most interesting and best established examples is that of the wild sheep. M. Severtzoff and Prof. Nassonov, after studying the distribution and structure of the wild sheep, have both come to the conclusion, that this genus migrated from High Asia to America before the last glaciation; that in America they were driven south by the glaciation; that afterwards the American stock spread again to the north and gave origin to another group; and finally that this new group spread back into the Asian continent and occupied eastern Siberia, so that the present Siberian sheep stand, geographically and structurally, in no close relation with the sheep of High Asia. It may be mentioned that a similar course of migration is now admitted for certain human tribes.

ON THE MECHANISM OF X-RAY SCATTERING

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It is now well known that whereas the usual form of the wave theory is inadequate to account for the alteration of the wave-length of X-rays when scattered, this may be explained simply on the hypothesis that the X-rays consist of quanta, each of which interacts with an individual electron. Strong support of this hypothesis is found in the discovery that there appear electrons each with about the momentum it should have acquired if it had deflected an X-ray quantum. That these electrons are associated with the scattered X-rays is evident from the fact that there exists on the average about one such electron for each quantum of scattered X-rays. Nevertheless the great weight of the evidence from many sources for the wave theory of radiation makes the acceptance of this radiation quantum hypothesis difficult.

It is improbable that the idea of spreading waves of radiant energy can be successfully reconciled with these studies of X-ray scattering without abandoning the principles of the conservation of energy and momentum. However, by making this bold step, Bohr, Kramers and Slater have apparently been able to incorporate them within the general scheme of the wave theory.

While discussing this matter in November 1923, Professor Swann called the attention of Professor Bohr and myself to the fact that a crucial test between the two theories could be made if it were possible to detect simultaneously a scattered quantum and the recoiling electron associated with it. For on any spreading wave theory, including that of Bohr, Kramers and Slater, there should be no correlation between the direction of ejection of the recoil electron and that of the effect of the scattered quantum. On the quantum view, however, if the recoiling electron is ejected at an angle θ with the incident X-ray beam, the scattered quantum should appear at an angle ϕ such that

$$\tan \frac{1}{2}\phi = -\frac{1}{1+\alpha}\cot\theta,$$

where $\alpha \equiv h/mc\lambda$.

Two methods of making this test suggest themselves. One is to use two point counters of the type developed by Geiger, Kovarik and others, one to receive the recoil electron and the other to catch the scattered quantum. The quantum hypothesis would predict simultaneous impulses in both chambers when set at the proper angles. This method has been

employed by Mr. R. D. Bennett in our laboratory, and has recently been suggested also by Bothe and Geiger[1] as a means of testing Bohr's theory.

The second method, which Mr. Simon and I have been using, is to take cloud expansion photographs showing simultaneously a recoil electron and a secondary β-ray track produced by the scattered X-ray quantum. To increase the probability of emission of a β-ray by a scattered quantum, diaphragms of thin lead foil are placed inside the chamber. Thus about 1 in 50 of the recoil electrons is found to have a secondary β particle associated with it.

Both methods have yielded provisional results. Mr. Bennett has connected each of a pair of head phones with one of the counting chambers

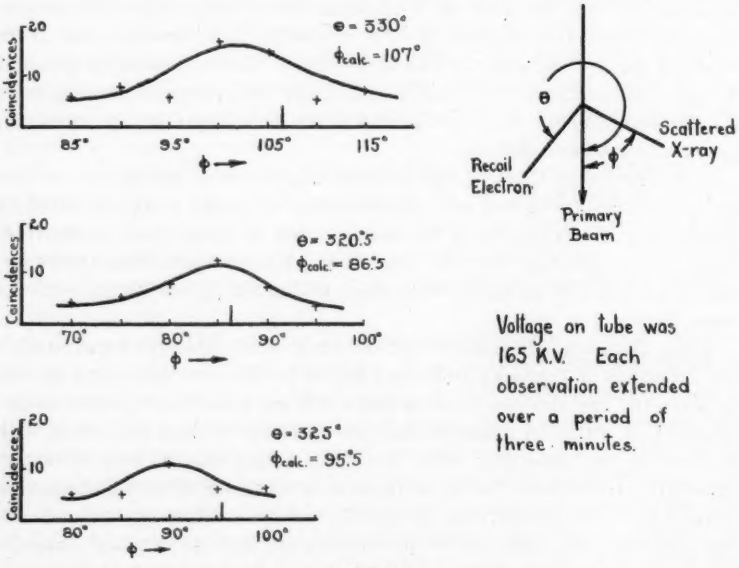

FIGURE 1

through a 3 stage amplifier, and has listened for simultaneous impulses in the two phones. He uses a thin strip of mica to scatter the X-rays, and places the counting chambers inside an evacuated vessel to avoid effects due to the curving of recoil electrons in air. His results for three different settings of the electron counter are shown in Figure 1. The fact that the coincidences are more frequent when the quantum counter is near the theoretical angle is in support of the quantum theory. Mr. Bennett hopes to be able to publish soon results obtained by a recording method, to avoid the uncertainties inherent in auditory observations of this type.

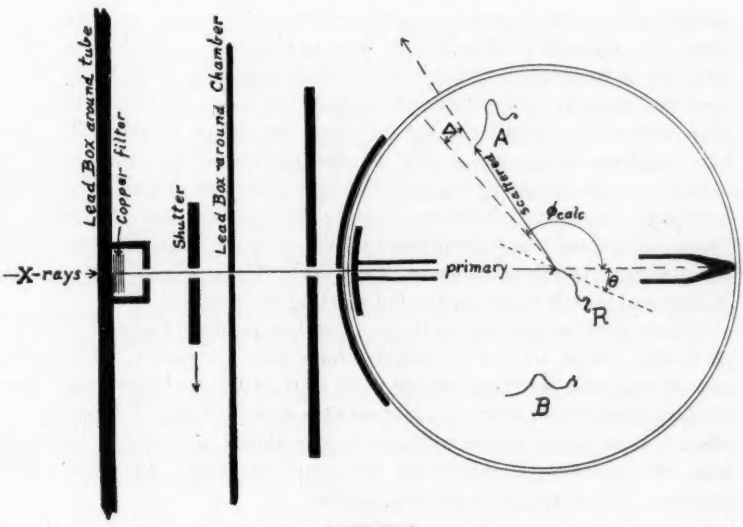

FIGURE 2

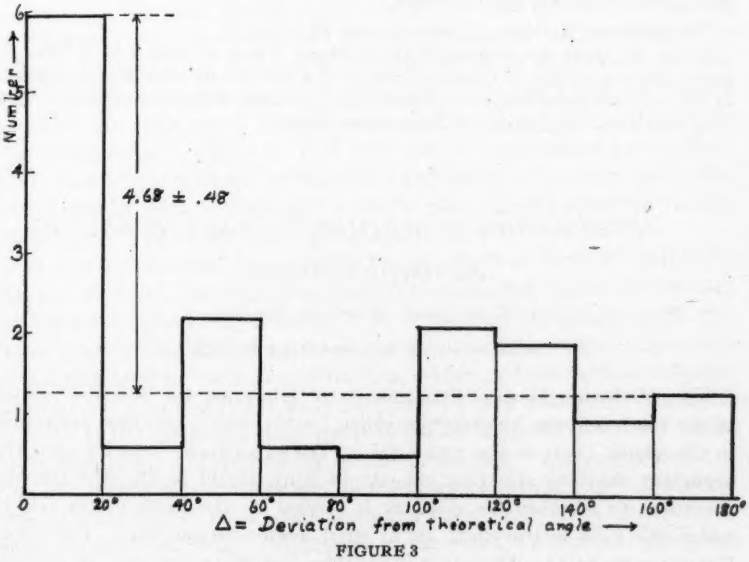

FIGURE 3

Mr. Simon and I have taken about 750 stereoscopic cloud expansion photographs, using apparatus such as that shown diagrammatically in figure 2. Our chief difficulty has been to eliminate stray X-rays, which give rise to meaningless tracks often indistinguishable from those due to scattered quanta. The results of the last 350 plates, in which these rays were reduced to a relatively low intensity, are shown in figure 3. Here I have plotted the deviation Δ of the observed tracks from the theoretical angle. On the spreading wave theory, the values of Δ should be nearly uniformly distributed between 0 and 180 degrees. On the quantum theory they should be concentrated near 0, as is obviously the case. The occurrence of tracks at other angles is explicable as due in part to stray X-rays and in part to the method of plotting the results.

A more detailed account of the work will be published when further experiments, which are now in progress have been completed. The results already obtained, however, permit us to state, with very little uncertainty, that the direction in which a quantum of scattered X-rays can produce an effect is determined at the moment it is scattered, and can be predicted from the direction of motion of the recoil electrons. In other words, *scattered X-rays proceed in directed quanta.*

It is possible to clothe this statement in the language of the wave-theory, if we keep in mind that a wave with a single quantum of energy can produce an effect in only one direction.[2]

[1] W. Bothe and H. Geiger, *Zeitschr. Physik.*, **26** (1924) 44.

[2] Since this paper was read before the Academy, I have received a letter from Dr. Bothe informing me that H. Geiger and he have also observed the coincidences demanded by the quantum theory but contrary to the theory of Bohr, Kramers and Slater. Their work will be described soon in *die Naturwissens haften.*

ETHER-DRIFT EXPERIMENTS AT MOUNT WILSON

By Dayton C. Miller

Case School of Applied Science

Read before the Academy April 28, 1925

The Michelson-Morley Experiment to determine the relative motion of the earth and the luminiferous ether, "ether drift," was first performed in Cleveland, Ohio, in the year 1887.[1] The experiment is based upon the argument that the apparent velocity of light should be slightly different according to whether the observer is carried by the earth in the line in which the light is travelling, or at right angles to this line. The interferometer devised by Michelson is capable of showing very minute changes in the relative velocities of two beams of light. Simple theory shows that

if the velocity of the earth's motion were directly effective in observations made with the interferometer, the apparent velocities of the light going in the direction of the earth's motion, and of that at right angles to this direction, would differ in the ratio of the square of the velocity of the earth to the velocity of light. The actual motion of the earth in space is the resultant of its oribital motion and of the motion of translation of the solar system as a whole. The latter component motion being unknown, it is impossible to predict any "expected" relative motion of the earth and ether. The interferometer is affected only by that component of the earth's total motion which lies in the horizontal plane of the apparatus. This plane is perpendicular to the radius of the earth at the location of the observatory, and because of the diurnal rotation and the annual revolution of the earth, this plane is continually changing with respect to space in general.

Michelson and Morley floated the interferometer on mercury, so that the instrument could be turned to all azimuths of the horizontal plane of observation in the effort to find the direction of the presumed ether-drift. A drift perpendicular to the plane of the interferometer would produce no effect whatever. The rotation of the earth on its axis would cause the plane of the interferometer to move as around the surface of a cone, and thus to take many different space orientations. Therefore the apparent azimuth of the drift should change with the time of observation. In November 1887, Michelson and Morley announced their conclusion as follows: "Considering the motion of the earth in its orbit only . . . the observations show that the relative motion of the earth and the ether is probably less than one-sixth the earth's orbital velocity and certainly less than one-fourth." (That is, it is less than $7\frac{1}{2}$ kilometers per second.) This result was considered by many as a null result, often called a negative result, and by some was thought to throw grave doubts upon the validity of the hypothesis of the luminiferous ether.

At the International Congress of Physics, held in Paris in 1900, Lord Kelvin expressed the conviction that the experiment should be repeated with a more sensitive apparatus. The writer in collaboration with Professor Morley constructed an interferometer about four times as sensitive as the one used in the first experiments, having a light path of 224 feet, equal to about 150,000,000 wave-lengths. In this instrument a relative velocity of the earth and ether equal to the earth's orbital velocity would be indicated by a displacement of the interference fringes equal to 1.5 fringes. This apparatus was used in the basement of the Physical Laboratory of Case School of Applied Science in Cleveland, observations being made in 1904 and 1905.[2] The result of these observations was published as follows: "We may therefore declare that the experiment shows that if the ether near the apparatus did not move with it, the difference in velocity

was less than 3.5 kilometers per second, unless the effect on the materials annulled the effect sought. Some have thought that this experiment only proves that the ether in a certain basement-room is carried along with it. We desire therefore to place the apparatus on a hill to see if an effect can be there detected."

In the autumn of 1905, Morley and Miller removed this interferometer from the laboratory basement to a site on Euclid Heights, Cleveland, at an altitude about 300 feet above Lake Erie, and free from obstruction of buildings. Five sets of observations were made in 1905 and 1906, which gave a definite positive effect of about $^1/_{10}$ of the then "expected" drift. There was a suspicion that this might be due to a temperature effect, though there was no direct evidence of this. A plan was made for putting this surmise to the test after a summer's vacation. We had erected the interferometer on land owned by a friend; during our vacation absence, the land was sold and the new owner ordered the immediate removal of the interferometer.

It seemed desirable that further observations should be carried out at a much higher altitude, but numerous causes prevented the immediate resumption of observations.

Through the kindness of President Merriam of the Carnegie Institution of Washington and of Directors Hale and Adams the experiments were resumed by the writer in March and April 1921, at the Mount Wilson Observatory in California where the elevation above sea level is about 6000 feet. The apparatus was substantially the same as that used by Morley and Miller in 1904, 1905 and 1906.

At the Mount Wilson station, about 5000 single measures of the ether-drift have been made at various times of day and night. These have been reduced in 204 different sets, each set consisting of observations made within one hour's time. The observations correspond to four different epochs of the year, as follows: (I) April 15, 1921, 117 sets of observations; (II) December 8, 1921, 42 sets; (III) September 5, 1924, 10 sets; and (IV) April 1, 1925, 35 sets.

I. Between April 9 and April 21, 1921, over 1300 single measures of the ether-drift were made. These were recorded in 117 sets of measures. By combining the sets made at nearly the same hour on different days, the number was reduced to six principal observations. Each observation gives for a specified time the magnitude of the displacement of the interference fringes, together with the azimuth of the line of sight in which the displacement is a maximum. This is equivalent to saying that the observations give directly the apparent relative motion of the earth and ether in kilometers per second, together with the direction of the drift. Since the light in the interferometer travels along one arm and back again, the observation indicates only the line in which the drift occurs, but does

not indicate whether it is in the positive or negative direction. The only criterion for determining the direction in the given line is the consistency of observations made at different epochs.

Figure 1 shows the observations of April 1921, in which the arrows indicate the observed direction and the magnitude of the drift, the arrows being located around a circle at points corresponding to the sidereal times of observation.

In order to eliminate any periodic effect of radiant heat, the metal parts of the interferometer were completely covered with large plates of cork about one inch thick. About fifty sets of observations were made under these conditions. The periodic displacement of the fringes due to the drift was the same with the cork in place, as it was without the cork.

II. In the summer of 1921, the steel frame of the interferometer was

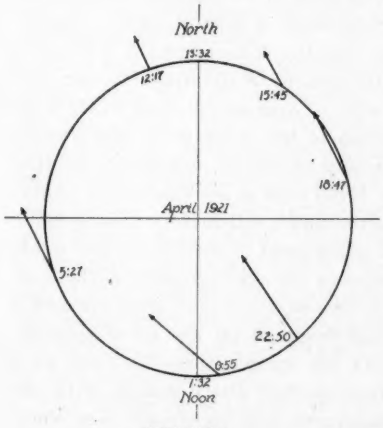

FIGURE 1

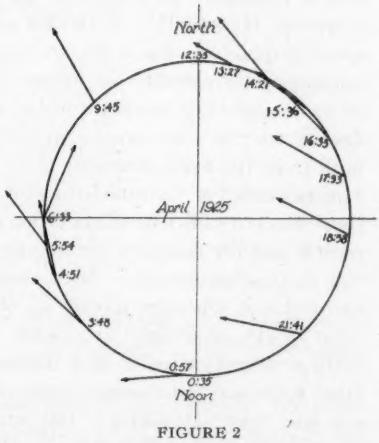

FIGURE 2

dismounted. A base of one piece of concrete, reinforced with brass rods, was cast in place on the mercury float. New supports of aluminum or brass were made for the optical parts. The entire apparatus was free from magnetic effects, and any possible effects of heat were much reduced.

From December 4 to December 11, 1921, about 900 single determinations of the drift were made, in 42 sets of observations. The results with this non-magnetic interferometer show a positive effect, as of an ether drift, of exactly the same magnitude and direction as was obtained at the first epoch in April 1921.

Many variations of incidental conditions were tried at this epoch. Observations were made with the centering pin tight in its socket, and then loose; with rotation of the interferometer clock-wise and counter clock-wise; with a rapid rotation of one turn in 40 seconds, and a slow rotation of

one turn in 85 seconds; with a heavy weight added first to the telescope arm of the main frame and then to the lamp arm; with the float extremely out of level because loaded first in one quadrant and then in the next quadrant; with the recording assistant walking round in different quadrants, and standing in different portions of the house, near to and far from the apparatus. The results of the observations were not affected by any of these changes.

The entire apparatus was returned to the laboratory at Cleveland; during the years 1922 and 1923, many trials were made under various conditions which could be controlled, and with many modifications in the arrangement of parts of the apparatus. An arrangement of mirrors and prisms was made so that the source of light could be placed outside of the observing room, the light entering the rotating interferometer along the axis of rotation. This method has been used in the regular observations of epochs III and IV. A further arrangement of mirrors, rather complicated in practice, was tried, for observing the fringes from a stationary telescope; the necessity for frequent adjustment of the fringes in the field of view made this method impracticable. Experiments were made with devices for the photographic registration of the positions of the fringes, both from the fixed observing station, and by means of a motion-picture camera carried on the interferometer. Even with an arc light as the source, there was not sufficient illumination to produce a satisfactory photographic record, and the necessity for frequent adjustment of the fringes also made this method unsuitable. After abandoning the photographic method, an astronomical telescope having an objective of five inches aperture and a focal length of seventy-five inches was mounted on the interferometer. With a magnification of fifty diameters, the fringes were observable on a large scale and with ample illumination, so that direct reading with the eye was very satisfactory; this arrangement has been used ever since. Trials were made with various sources of light; with electric arc and incandescent lamps, the mercury arc, acetylene lamp, and also with sun light. The interchange between sun light and laboratory sources in no way altered the results. The final choice for the stationary source placed outside of the interferometer room (or house, on the mountain) was a large acetylene lamp of the kind commonly used for automobile head-lights.

An extended series of experiments was made to determine the influence of inequality of temperature in the interferometer room, and of radiant heat falling on the interferometer. Several electric heaters were used, of the type having a heated coil near the focus of a concave reflector. Inequalities in the temperature of the room caused a slow but steady drifting of the fringe system to one side, but caused no periodic displacement. Even when two of the heaters were placed at a distance of three feet from the interferometer as it rotated, and were turned to throw the heat directly

on the uncovered steel frame, there was no periodic effect that was measureable. When the heaters were turned on to the light-path which had a covering of glass, a periodic effect could be obtained only when the glass was covered with opaque material in a very non-symmetrical manner, as when one arm of the interferometer was completely protected by a covering of corrugated paper-board while the other arms were unprotected. These experiments proved that under the conditions of actual observation, the periodic displacement could not possibly be produced by temperature effects.

As an indication of the reliability of the readings made with the interferometer, a series of observations are shown in figure 3, taken in the laboratory at Cleveland in July 1924. Two of these sets were made with sunlight as the source, the third and fourth lines from the top. The uncertainty of reading is less than 0.01 of a fringe, and the displacement is zero.

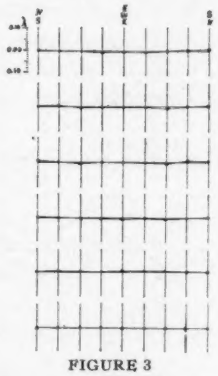

FIGURE 3

The investigation in the laboratory demonstrated that the full period effect mentioned in the preliminary report on the Mount Wilson observations,[3] is a necessary geometrical result of the adjustment of the mirrors in producing fringes of finite width. Under poor temperature conditions, such as existed at Mount Wilson in April 1921, it was necessary to use narrow fringes, in which case the full period effect is relatively large; as the width of fringes is increased, this effect decreases and vanishes only when fringes of infinite width are used, as is presumed is the simple theory of the experiment.

III. After the conclusion of the experiments just described, the interferometer was taken again to Mount Wilson. In 1921, the apparatus had been located on the very edge of a deep canyon; it was feared that the air currents up and down the face of the canyon might produce a disturbance, and also that the unsymmetrical distribution of the rock of the mountain itself might be undesirable. In August 1924, a new site was chosen, on a very slightly rounded knoll, removed from the canyons. The interferometer house was erected with its orientation, as regards the ridge of the roof and the location of the door, changed by 90° from that of 1921. In every detail, the interferometer was the same as had been used in Cleveland in July 1924. On September 4, 5 and 6, 1924, 275 measures of the drift were made, in 10 sets of observations. The actual observations are plotted in Fig. 4. This shows a positive displacement of the fringes, in contrast with the small result obtained in Cleveland, shown in Fig. 3. The corresponding ether-drift is consistent in direction and magnitude

with that previously obtained at Mount Wilson. Part of these observations were made with the glass case over the light path, covered with corrugated paper which had been found in the Cleveland experiments to exclude all effects of radiant heat. This covering produced no effect whatever, showing that there was no disturbance of this kind.

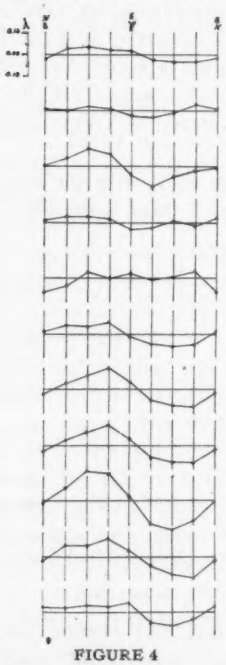

FIGURE 4

IV. Observations on Mount Wilson were resumed on March 27, 1925, and continued until April 9, during which time 1600 measures of the drift, in 35 sets, were made. The interferometer was as used in September 1924, excepting that new mirrors were provided for bringing in the acetylene light. During the first half of the observations for this epoch, the acetylene lamp was stationary and outside of the house; during the last half of the series, the lamp was placed on the cover of the interferometer, near the axis of rotation, which simplified some of the adjustments. There was not the slightest difference in the results obtained with the two positions of the light. The position of the assistant in the house was changed by a quadrant, with no effect on the result.

Throughout epoch IV the conditions of observation were exceptionally good; some of the time there was a fog, which rendered the temperature very uniform. Four precision thermometers were hung on the outside walls of the house; on numerous occasions the extreme variation of temperature was not more than 0.1°, and usually it was less than 0.4°; however, a variation of several degrees, while causing a constant drift of the fringe system, did not change the periodic displacement either in azimuth or magnitude.

The observations made at about the same time of day (or night) being combined, the 35 sets have been averaged in thirteen groups, and the resulting ether-drifts are shown in figure 2. The agreement between the results for 1921 and 1925 is striking, especially when it is recalled that the interferometer has been rebuilt as to its details, has a different system of illumination and observation, and has been changed to a new site, in a house differently oriented, and that many variations in observational procedure have been tried.

The interferometer readings being plotted, give directly, by harmonic analysis (carried out with the mechanical harmonic analyzer) the azimuth and magnitude of the ether-drift. There are no corrections of any kind

to be applied to the observed values. In the work so far, every reading of the drift made at Mount Wilson has been included at its full value. No observation has been omitted because it seemed to be poor, and no "weights" have been applied to reduce the influence on the results, since no assumption has been made as to the expected result. It may be added that while the readings are being made, neither the observer nor the recorder can form the slightest idea as to whether any periodicity is present, much less as to the direction or amount of such periodicity.

The test of these observations is whether they lead to a rational and wholly consistent indication of a constant motion of the solar system in space, combined with the orbital motion of the earth and the daily rotation on its axis. There is a specific relation for a given latitude between the observed azimuth of drift and the sidereal time of observation. Observations at different sidereal times should show different azimuths and all observations at the same sidereal time should show the same azimuth, for

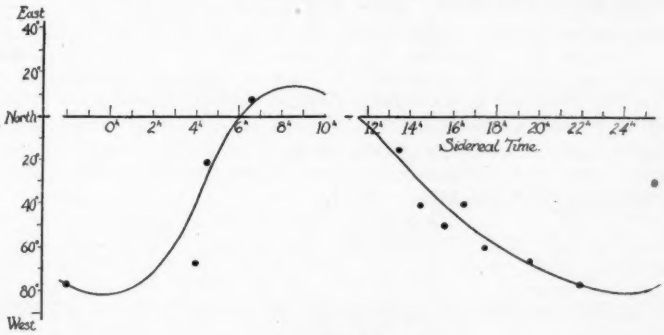

FIGURE 5

a given epoch. The 1600 observations of the ether-drift made in April 1925, consisting of 35 sets made on different days, have been combined into eleven sets at different sidereal times, and are charted in azimuth with respect to sidereal time in figure 5. The curve shown has been drawn arbitrarily to indicate that there is a definite relation. This curve is of the kind that would correspond to some definite direction and velocity of ether drift. The observations for the other three epochs, while not so numerous, give curves wholly consistent with this one.

It need hardly be said that the determination of the absolute motion of the solar system from such interferometer observations, is one of great complexity. Prof. J. J. Nassau of the Department of Mathematics and Astronomy of Case School of Applied Science, and Dr. G. Strömberg of the staff of the Mount Wilson Observatory, have given very great assistance in the mathematical analysis, and have developed solutions of various parts

of the problem, and also a complete least-squares solution of the general problem. A definitive numerical calculation will require several months of continuous work, and is now in progress.

The ether-drift experiments at Mount Wilson during the last four years, 1921 to 1925, lead to the conclusion that there is a relative motion of the earth and the ether at this Observatory, of approximately nine kilometers per second, being about one-third of the orbital velocity of the earth. By comparison with the earlier Cleveland observations, this suggests a partial drag of the ether by the earth, which decreases with altitude. It is believed that a reconsideration of the Cleveland observations, from this point of view, will show that they are in accordance with this presumption and will lead to the conclusion that the Michelson-Morley Experiment does not give a true zero result. A complete calculation of the observations, now in progress, together with further experiments to be made in the immediate future, should give definite indications regarding the absolute motion of the solar system in space.

[1] Michelson and Morley, *Relative Motion or the Earth and the Luminiferous Ether, Amer. J. Sci.*, **34**, 333 (1887); *Phil. Mag.*, **24**, 449 (1887); *J. Physique*, **7**, 444 (1888).

[2] Morley and Miller, *An Experiment to Detect the FitzGerald-Lorentz Effect, Phil. Mag.*, **9**, 680 (1905); *On the Theory of Experiments to Detect Aberrations of the Second Degree, Phil. Mag.*, **9**, 669 (1905). *An Experiment to Detect the FitzGerald-Lorentz Effect, Proc. Amer. Acad. Arts Sci.*, **41**, 321 (1905).

[3] Miller, *Ether-Drift Experiments at Mount Wilson Observatory, Physic. Rev.*, **19**, 407 (1922); *Science*, **55**, 496 (1922).

THE INTENSITIES OF LINES IN MULTIPLETS. I. THEORY

By Henry Norris Russell

Mount Wilson Observatory, Carnegie Institution of Washington

Communicated May 8, 1925

It appears probable that the relative intensities of spectral lines belonging to the same multiplet group are determined by quantum conditions. This note presents formulae recently developed by the writer, which closely represent all the available data.

1. *Notation.*—From the standpoint of multiplet structure, a spectroscopic term may be completely specified by three integers, one of which defines the system to which the term belongs, the second the series, and the third distinguishes between the components of a multiple term. The most natural definition of these integers appears to be that which makes the first equal to the maximum number of components possessed by any term of the system: the second to the maximum number of components

which a term of the given series can have, and the third to the number of components into which the individual level is divided in the Zeeman effect.

Following Landé,[1] these integers may be called $2R$, $2K$ and $2J$ (in Sommerfeld's[2] notation they are $2j_s + 1$, $2j_a + 1$ and $2j + 1$). $2R$ has the values 1, 2, 3 for the singlet, doublet, triplet, systems. $2K$ is 1, 3, 5, 7 for the S, P, D, F terms, so that $K = k - 1/2$ when k is the ordinary azimuthal quantum number. J is equal to the inner quantum number j (as commonly used) for the systems of even multiplicity but to $j + 1/2$ for odd multiplicities. When $K > R$, J ranges by steps of a unit from $K - R + 1/2$ to $K + R - 1/2$, and the "permanent" number of components $2R$ occurs; but, if $K < R$, J ranges from $R - K + 1/2$ to $R + K - 1/2$ and there are $2K$ components ("smaller" multiplicity).

Under ordinary conditions spectral lines correspond only to transitions for which ΔJ is 0 or ± 1. ΔR is zero in most cases, but ± 1 for intersystem combinations. ΔK is confined to the values ± 1 in the simplest spectra, but is often zero in the more complex spectra, and occasionally ± 2. (Indeed, the writer has found a few faint lines of Ti and Fe^+ for which $\Delta K = \pm 3$.)

Consider now a multiplet involving terms of the same system ($\Delta R = 0$), and let $K_1 + K_2 = 2K$. There are three principal types of structure, illustrated here by examples from the quintet system:

ORDINARY

$\Delta K = 1 \quad K > R$

$2K_1 = 5. \quad 2K_2 = 7$ (DF)

$2J$	11	9	7	5	3
9	x_4	y_3	z_2		
7		x_2	y_1	z_0	
5			x_0	y_{-1}	z_{-2}
3				x_{-2}	y_{-3}
1					x_{-4}

RHOMBOID

$\Delta K = 1 \quad K < R$

$2K_1 = 3. \quad 2K_2 = 5$ (PD)

$2J$	9	7	5	3	1
7	x_3	y_2	z_1		
5		x_1	y_0	z_{-1}	
3			x_{-1}	y_{-2}	z_{-3}

SYMMETRICAL

$\Delta K = 0$

$2K_1 = 2K_2 = 7$ (FF')

$2J$	11	9	7	5	3
11	x_4	y_3			
9	y_3	x_2	y_1		
7		y_1	x_0	y_{-1}	
5			y_{-1}	x_{-2}	y_{-3}
3				y_{-3}	x_{-4}

The "Rhomboid" groups have not previously received a special name (which may excuse the obvious one coined here for them). They correspond to terms of smaller multiplicity; the ordinary multiplets to permanent multiplicity. The symmetrical groups are similar in both instances. When $K = R$ a special case arises, and the line for which $J_1 + J_2$ has the smallest value is absent. When $2R$ is even, this happens because one term has only $2R - 1$ components; when $2R$ is odd, the missing line is excluded by Landé's "Zusatzregel," both values of j being zero.

The intensities of the various lines may now be represented by x, y, z, as above. The subscripts (which are chosen for the sake of symmetry) are the values of l, defined by the equation $l = J_1 - K_1 + J_2 - K_2 = J_1 + J_2 - 2K$ when $K > R$, or by $l = J_1 + J_2 - 2R$ when $R > K$.

2. *Qualitative Rules for Intensity.*—Qualitative study of the observed

line intensities leads to the following empirical rules (which are independent of the quantitative meaning of the arbitrary scales on which the intensities are estimated). Several of these rules are well known: the rest may be sufficiently verified from King's data, summarized below:

(1) The leading x (that is, the one with the greatest value of l) is always the strongest line in the multiplet, and the following x's decrease steadily in intensity.

(2) The y's are strongest near the middle of the sequence.

(3) The leading z is the weakest line in the multiplet (when the z's are present at all).

(4) When $K > R$, the y's average much fainter than the x's, and the z's then the y's (Sommerfeld's rule). These ratios decrease as K/R increases; and y/x is nearly the same for symmetrical multiplets as for the adjacent ordinary ones.

(5) When $K > R$, the fall of intensity along the x's is most conspicuous when $K-R$ is small. When $K-R = 0$ the last x is absent, and when $K-R = 1$ it is usually fainter than the adjacent y's. The increase in intensity along the z's is relatively small.

(6) In rhomboid groups the values of z increase rapidly, and the last line is often strong.

(7) In symmetrical groups, when $K < R$, the later values of x are small, and the next to the last line is sometimes almost or completely absent. (Meggers.)

(8) When $\Delta K = 1$ the values of y are substantially equal for lines equidistant from the two ends, but when $\Delta K = 0$ the later lines are fainter, especially when $K = R$.

3. *Quantitative Rules for Intensity.*—The quantitative measures of intensity (at present few, but rapidly increasing) conform, in all cases not complicated by self-reversal, to the fundamental *sum-rule.*[3] *The sum of the intensities of the lines which have a given initial (or final) level in common is proportional to the statistical weight of this level.* In other words, the sum of the intensities in any row or column is proportional to the corresponding value of $2J$. This rule is not sufficient to determine the intensities except when $R = 2$ and $\Delta K = 1$.

Further data may be supplied by the postulate that the intensities are expressible by general formulae, which satisfy the sum-rule, algebraically, for all values of J, K and R. The fictitious lines thus introduced may have any intensities, positive or negative; but the sum of the intensities of those fictitious lines which lie in the same row or column as any real line must obviously be zero. If it is assumed that the intensity of each of these lines is separately zero, certain roots of the functions which represent the general intensity are given, and their forms are partially determined by this *zero rule.*

Finally, the correspondence principle[4] applied to cases when K or R is great, suffices for an almost complete solution of the problem.

To satisfy the sum-rule, the sum of each column must be of the form $CJ_1 \Sigma J_2$ and for each row $CJ_2 \Sigma J_1$. Now for every term, whether of permanent or smaller multiplicity, $\Sigma J_1 = 4RK_1$ so that the sum S of the intensities for the whole multiplet is $16CR^2K_1K_2$, or $16CR^2K^2$ for symmetrical, and $4CR^2(4K^2-1)$ for ordinary and rhomboid groups. By setting $C = 4K^2-1$ in the first case, and $4K^2$ in the second the formula

$$S = 16R^2K^2(4K^2-1) \tag{1}$$

suffices for all groups.

4. *Ordinary Multiplets.*—Consider first the ordinary multiplets. The equations given by the sum-rule are easily found to be

$$\begin{aligned} x_l + y_{l+1} + z_{l+2} &= 8RK^2(2K-1)\ (2K+1+l) \\ x_l + y_{l-1} + z_{l-2} &= 8RK^2(2K+1)\ (2K-1+l) \end{aligned} \tag{2}$$

from which it follows that

$$y_{l-1} + z_l + z_{l-2} = y_{l+1} + z_{l+2} + z_l + 16RK^2l. \tag{3}$$

The last physically real values of x correspond to $l = \pm(2R-1)$, of y to $\pm(2R-2)$, and of z to $\pm(2R-3)$. The zero-rule shows that $y = 0$ when $l = \pm 2R$, and $z = 0$ when $l = \pm(2R-1)$, $l = \pm(2R+1)$.

Assuming this, it follows from (3) that for all values of l,

$$y_l + z_{l+1} + z_{l-1} = 4RK^2(4R^2-l^2) \tag{4}$$

whence, substituting in (2)

$$x_l - z_l = 4RK^2(8K^2 + 4Kl - 4R^2 + l^2 - 1). \tag{5}$$

The zero-rule shows that

$$z = \psi(l)\{(2R+1)^2-l^2\}\{(2R-1)^2-l^2\} \tag{6}$$

but does not define the function $\psi(l)$.

The correspondence principle may now be applied in detail. According to Sommerfeld and Heisenberg this demands that

$$x:y:z = (1+\cos\theta)^2 : 2(1-\cos^2\theta) : (1-\cos\theta)^2 \tag{7}$$

where θ is the angle between the vectors representing the angular momenta of the "light-electron" and of the whole atom.

The "vector model" of Sommerfeld[5] gives

$$\cos\theta = \frac{j^2 + j_a^2 - j_s^2}{2jj_a}$$

while that of Landé[6] gives

$$\cos\theta = \frac{J^2 + K^2 - R^2}{2JK}. \tag{8}$$

Ornstein and Burger[7] have discussed the first of these, and have shown that it leads to a formula similar to, but not identical with, equation (4) (which they obtain by substantially the same reasoning as above). Landé's equation, however, gives

$$\begin{aligned} x &= C\{(J+K)^2-R^2\}^2 \\ y &= 2C\{(J+K)^2-R^2\}\{R^2-(J-K)^2\} \\ z &= C\{R^2-(J-K)^2\}^2 \\ x+y+z &= 16CJ^2K^2 \end{aligned} \tag{9}$$

The factor C may be determined by setting $x + y + z$ equal to the value given by the sum-rule, taking the mean of the two equations (2), or, preferably, by setting the sum for the whole multiplet equal to (1).

Which of the two inner-quantum numbers of the terms concerned should be taken to define J is not prescribed by the correspondence principle, but it appears reasonable to take a symmetrical function of the two, since the initial and final states appear to be of equal importance in determining the intensities. The simplest such function is the mean. In this case $J = \frac{1}{2}(J_1 + J_2) = K + \frac{1}{2}l$ and (9) and (2) give

$$C = 2R\{1-(2Kl + l^2 + 1)/(2K + l)^2\}$$

whence (expanding in descending powers of K and neglecting $1/K$ except in z)

$$\begin{aligned} x &= R(32K^4 + 16K^3l - 16K^2R^2 + 4K^2l^2 - 8K^2 + 2R^4 - R^2l^2 + \tfrac{1}{8}l^4 + 2) \\ y &= R(4K^2 - R^2 + \tfrac{1}{4}l^2 + 1)(4R^2 - l^2) \\ z &= \tfrac{1}{8}R\left\{1 - \frac{l}{2K + l}\right\}(4R^2 - l^2)^2. \end{aligned} \tag{10}$$

These values satisfy the equations (4) and (5) almost exactly, the error of the approximation being only $1/(4K^2)$ of the whole amount.

These equations follow from the sum-rule alone and may be regarded as well established. The excellent agreement of (10) with them tells strongly in favor of Landé's vector model, as does also the fact that this model gives the correct interval-rule for the separations of the components.

For the final value of z, (6) and (10) give

$$z = \tfrac{1}{8}R\left(1 - \frac{l}{2K + l}\right)\{(4R^2 - l^2 - 1)^2 - 4l^2\} \tag{11}$$

whence by (4)

$$y = R\{4K^2 - (R^2 + \tfrac{1}{4}l^2 + 1)\}(4R^2 - l^2) + \ldots$$

The value of y now agrees exactly with (10) except for the term in $1/K$. That of x may be found from (5) and disagrees with (10) by $\dfrac{1}{8K^2}$ of the whole amount. In actual computation z is best found from (11) and x and y from (5) and (4).

Ornstein and Burger give for the z's the expression

$$J_{+1} = \frac{r\{j_z(j_z + 1) - (k - j)(k - j + 1)\}^2}{4j(j + 1)}$$

when j is the larger of the two inner-quantum numbers and k the smaller of the two azimuth quantum numbers involved. Reduced to the present notation this is

$$z = \frac{RK^2\{4R^2 - (l-1)^2\}^2}{2(2K + l)(2K + l + 2)}$$

This is zero when $l = 2R + 1$ and $1-2R$, but not when $l = 2R-1$ and $-1-2R$. This fact, combined with the asymmetry of the expression, appears to be to its disadvantage. The greatest discrepancy between its results and that of (11) is for $2R = 3$, $2K = 4$, $i = 0$ (triplet *PD*) when it gives 16/25 as great a value. Exact measures on such groups should settle which formula is best.

5. *Rhomboid Multiplets.*—When $R > K$ the sum-rule gives the equations

$$\begin{aligned} x_l + y_{l+1} + z_{l+2} &= 8RK^2(2K-1)(2R + 1 + l) \\ x_l + y_{l-1} + z_{l-2} &= 8RK^2(2K + 1)(2R-1 + l). \end{aligned} \tag{12}$$

The zero-rule indicates roots of the intensity-function when l is $-2K \pm 1$ for x, $\pm K$ for y and $2K \pm 1$ for z. Setting

$$\begin{aligned} x_l &= A(l + 2K + 1)(l + 2K-1) \\ y_l &= B(4K^2-l^2) \\ z_l &= C(l-2K + 1)(l-2K-1) \end{aligned}$$

it is easily found that if $A = 2KR\,(R + K)$, $B = 4KR^2$, $C = 2KR\,(R-K)$, the equations (12) are identically satisfied. When $R = K$ the resulting expressions for x and for y become identical with the second members of (5) and (4), while z vanishes. To complete the solution it suffices to add terms similar to (11) but with R and K interchanged, which gives the equations

$$\begin{aligned} x_l &= u_l + 2KR(R + K)(l + 2K + 1)(l + 2K-1) \\ y_l &= 4KR^2(4K^2-l^2) - (u_{l-1} + u_{l+1}) \\ z_l &= u_l + 2KR(R-K)(l-2K + 1)(l-2K-1) \\ u_l &= \tfrac{1}{8} K \left(1 - \frac{l}{2R + l}\right) \{(4K^2-l^2-1)^2 - 4l^2\}. \end{aligned} \tag{13}$$

When K is negligible in comparison with R these equations become, except for a constant factor, identical with those given by Hönl[8] for the intensities of the Zeeman components in the case when the number of magnetic levels is different in the two terms. (These equations, which were found three years ago by Heisenberg, but not published,[9] were also hit upon independently by the writer.) In this case $\Sigma x = \Sigma y = \Sigma z$, in accordance with the polarization-rule of Ornstein and Burger;[10] in the more general case $\Sigma x : \Sigma y : \Sigma z = R + K : R : R-K$ neglecting the small u terms.

The correspondence principle gives in the case when $J = R + \frac{1}{2} l$

$$\begin{aligned} x &= C(4R + 2K + l)^2(2K + l)^2 \\ y &= 2C\{(4R + l)^2 - 4K^2\}(4K^2-l^2) \\ z &= C(4R-2K + l)^2(2K-l)^2 \\ x + y + z &= 64CK^2(2R + l)^2 \end{aligned}$$

Comparing this with the mean of equations (12) it appears that

$$C = \frac{RK}{8R+4l} - \frac{R}{8(2R+l)^2}.$$

A more precise value of C may be derived from the condition that, when K is fixed and R increases without limit, the relative intensities approach those of a Zeeman pattern, for which $\Sigma x = \Sigma y = \Sigma z = \frac{1}{3} S$. This gives (neglecting all terms not containing R) $\Sigma y = 32CR^2\Sigma(4K^2 - l^2) = \frac{16}{3} R^2K^2(4K^2-1)$. In the summation, l has the values $\pm 1, \pm 3, \ldots \pm (2K-2)$; its value is easily found to be $\frac{4}{3} K(4K^2-1)$. Hence $C = \frac{1}{8} K$ when R is very great.

This indicates that the second term in the expression previously found should be dropped, which is legitimate as it is of the order $\frac{1}{2RK}$ with respect to the first. This gives

$$\begin{aligned} x &= \{2RK(R+K) + \tfrac{1}{8} K(2K-l)^2\}(2K+l)^2 \\ y &= \{4KR^2 - \tfrac{1}{4} K(4K^2-l^2)\}(4K^2-l^2) \\ z &= \{2RK(R-K) + \tfrac{1}{8} K(2K+l)^2\}(2K-l)^2 \end{aligned} \qquad (14)$$

which agree exactly with (13) as far as they go. The correspondence principle, as here developed, is therefore sufficient to define completely all the terms in the expressions for x, y, z, which are of the highest order in R, K, l. The terms of lower orders must be determined by the zero-rule.

6. *Symmetrical Multiplets.*—For these, when $K > R$, the sum-rule gives

$$y_{l-1} + x_l + y_{l+1} = 4RK(4K^2-1)(2K+l) \qquad (15)$$

(the number of equations being halved, on account of the symmetry). The zero-rule gives

$$y = \psi(l)(4R^2-l^2).$$

The correspondence principle demands in this case that $2y : x = \sin^2\theta : \cos^2\theta$ whence

$$\begin{aligned} x &= C(J^2 + K^2 - R^2)^2 \\ 2y &= C\{4K^2R^2 - (J^2-K^2-R^2)^2\} \\ x + 2y &= 4CJ^2K^2 \end{aligned}$$

Here $J = K + \frac{1}{2} l$ and so

$$C = \frac{4R(4K^2-1)}{K(2K+l)}$$

$$y = \tfrac{1}{2} R(4K^2-1)\left\{2 - \frac{(4R^2-l^2)}{4K(2K+l)}\right\}(4R^2-l^2).$$

These values may be expected to be correct as regards the terms of highest degree in R, K, l, but to need correction in the terms involving lower powers of these quantities.

When $R > K$ the sum-rule gives

$$y_{l-1} + x_l + y_{l+1} = 4RK(4K^2-1)(2R + l). \qquad (16)$$

The zero-rule

$$y = \psi(l)(4K^2-l^2)$$

and the correspondence principle, with $J = R + \frac{1}{2} l$

$$C = \frac{4R(4K^2-1)}{K(2R+l)}$$

$$y = C(4K^2-l^2)(R^2 + \tfrac{1}{2} Rl - \tfrac{1}{4} K^2 + \tfrac{1}{16} l^2).$$

The condition that, when K is fixed and R increases without limit, the relative intensities approach those of a Zeeman pattern, gives in this case $C = 4K$ when R is great. The resulting Zeeman patterns are exactly the same as those deduced by Hönl (loc. cit.).

This limiting value may be obtained by settling $4K^2$ instead of $4K^2-1$ in the expression previously found, which gives

$$C = \frac{8RK}{2R+l}$$

$$y_l = 2RK\left(2R - \frac{4K^2-l^2}{4(2R+l)}\right)(4K^2-l^2). \qquad (17)$$

Continuity demands the same correction when $K > R$, giving

$$y_l = 2RK\left(2K - \frac{4R^2 - l^2}{4(2K+l)}\right)(4R^2 - l^2). \qquad (18)$$

The equations (15), (16), (17) and (18) afford a solution of the problem, derived entirely from general principles (as is also the case for the multiplets of other types).

In the case when $R = K$, they give

$$y_l = \tfrac{1}{2} K^2(6K + l)(4K^2 - l^2)$$

for which it follows that

$$\begin{aligned} x_l &= K^2(8K^3-2K + (12K^2-1)\, l + 6Kl^2 + l^3) \\ &= K^2(l + 2K-1)(l + 2K)(l + 2K + 1). \end{aligned} \qquad (19)$$

For the last line of the diagonal, $l = 1-2K$, and $x = 0$. This is the line corresponding to the transition between two levels of inner quantum

number zero (in the common notation) and excluded by Landé's rule. This rule appears therefore to be a consequence of the combination principle, as Burger and Hönl have already shown in the analogous case of the Zeeman patterns.

A further test of the formula, and a severe one, is found in the symmetrical multiplets for which $R > K$. In the DD' group of the sextet system, the next line to the end is very faint, and in the corresponding septet group it is absent. Both phenomena are predicted by the present theory.

[1] The references will be found all together at the end of the second paper on p. 328.

THE INTENSITIES OF LINES IN MULTIPLETS. II. OBSERVED DATA

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The theory developed in the preceding paper is here compared with the observed data.

(*a*) *Precise Measures.*—Very few precise measures of intensity have yet been published for multiplets of the triplet and higher systems—which alone are of importance to test the theory, as the sum-rule suffices for the doublets. To save space they are printed below with the x's, y's and z's in the same line, separated by semi-colons, rather than in the ordinary multiplet arrangement. In the symmetrical groups, the means of the observed values have been taken for the lines which should theoretically be equal (the observed differences being within the observational errors). The computed values are adjusted so that their sum equals that of the observed intensities for the whole multiplet.

Triplet System[11]

PD	obs.	100,	54,	25;	19,	18;	<1
	comp.	101,	54.5	24	18	18	1.2
DF	*o*.	100,	69.	48;	9.	9;	<1
	c.	101	70	47	8.7	8.7	0.2
DD'	*o*.	100.	56.	37;	14.5	13	
	c.	102	57	37	13	12	

Quintet System[12]

DD'	*o*.	100,	46,	19.5,	3,	0;	24,	34,	24,	13.7
	c.	108	50	18	3.6	0	22	29	25	14.4

The agreement is, in all cases, within the accuracy with which the observations satisfy the sum-rule.

(*b*) *King's Estimates.*—King, in his study of the variations of spectra with temperature, has published[13] long series of very careful estimates of the intensities of thousands of lines, in arc and furnace spectra, on an empirical but apparently a remarkably homogeneous scale. According to a verbal communication from Dr. King, this scale was intended to represent the actual intensities of the lines, and allowance was made for the probable effects of self-reversal.

Comparison with the computed data for groups belonging to the doublet and triplet systems, where there can be little or no doubt about the theory, shows that King's estimated "intensities" are very nearly proportional to the square roots of the actual intensities. The agreement is so close that these estimates, especially when averages for several multiplets are available, are clearly almost as valuable as actual measures, when once the significance of the empirical scale has been found. The analogy with estimates of stellar magnitude will occur at once to the astronomer.

In this way a great mass of data becomes available, covering almost all the complex spectra which have been analyzed, and including representatives of practically every known type of multiplet. Doublets are found in Sc, Ca^+, Sr^+, Ba^+, Ti^+; triplets in Ca, Sr, Ba, Ti, Fe, Sc^+; quartets in V, Mn, Co, Ti^+; quintets in Ti, Cr, Fe, V^+; sextets in V, Mn, Fe^+; septets in Cr, Fe, Mn^+; and octets in Mn. The multiplets of each type were collected from all these spectra, and the intensities, as observed in the arc spectrum, were reduced to a common standard by multiplication by factors such that the sum of the intensities for the whole multiplet became 100 *R;* that is, 100 for doublets, 150 for triplets, etc. Thus reduced, the relative intensities for multiplets from different spectra were found to be in good agreement. A few discordant cases were obviously due to varying sensibility of the plates (as when some lines lay in the deep red). A very few groups showing quite abnormal intensities were rejected, since it is quite possible that in complex spectra like those of Ti or Co, groups differing from the normal structure may exist.

The resulting mean intensities for the various multiplets are given below. The type of multiplet is given at the left; the number of groups of this type combined into the observed means is in parentheses below it. For each type the first line, *I*, gives the actual intensities, computed by the formulae of this paper; the second, *S*, the square roots of these intensities, and the third, *O*, King's observed values. The sums of *S* and *O* have been adjusted to equal 100 *R;* those for *I* have various values, dependent upon the convenience of calculation. For the doublet and triplet systems, the intensities are expressed as ratios of integers; for the other systems this is usually not the case, and the computed values are rounded off to

the nearest integer, or the nearest tenth, if they are small. The values of S are usually given to the nearest integer. A few groups of theoretical interest, for which no observations exist, are added for completeness.

Doublet System

SP	*I*	4,	2;		*PP'*	*I*	10,	4;	2
(7)	*S*	59	41			*S*	39	25	18
	O	56	44			*O*			
PD	*I*	18,	10;	2	*DD'*	*I*	28,	18;	2
(9)	*S*	48	36	16	(6)	*S*	44	35	11
	O	50	38	12		*O*	42	36	11
DF	*I*	40,	28;	2	*FF'*	*I*	54,	40;	2
(5)	*S*	48	41	11	(2)	*S*	45	38	8
	O	50	39	11		*O*	48	40	6
FG	*I*	70,	56;	2	*GG'*	*I*	88,	70;	2
(2)	*S*	49	43	8	(2)	*S*	46	40	7
	O	49	43	8		*O*	48	40	6
GH	*I*	108,	88;	2	*HH'*	*I*	130,	108;	2
(2)	*S*	49	44	7		*S*	46	43	6
	O	49	44	2?		*O*			

Triplet System

SP	*I*	5,	3,	1			*PP'*	*I*	30,	6,	0;	10, 8
(7)	*S*	68	52	30			(10)	*S*	41	19	0	24 21
	O	69	54	27				*O*	44	18	0	24 20
PD	*I*	84,	45,	20;	15,	15; 1	*DD'*	*I*	448,	250,	162;	56, 54
(15)	*S*	48	35	23	19	19 5	(5)	*S*	40	30	24	14 14
	O	47	36	25	19	19 4		*O*	41	30	25	14 13
DF	*I*	405,	280,	189;	35,	35; 1	*FF'*	*I*	1215,	847,	640;	81, 80
(13)	*S*	48	40	32	14	14 2	(8)	*S*	42	35	30	11 10^+
	O	48	36	35	14	14 3		*O*	42	35	29	11 11
FG	*I*	1232,	945,	720;	63,	63; 1	*GG'*	*I*	4224,	3249,	2625;	176, 175
(9)	*S*	48	42	37	11	11 1.3	(1)	*S*	43	38	34	9 9
	O	50	40	36	10	10 ?		*O*	46	38	30	9 9
GH	*I*	2925,	2376,	1925;	99,	99; 1	*HH'*	*I*	11375,	9251,	7776;	325, 324
(4)	*S*	48	42	37	9	9 1	(1)	*S*	44	40	36	7.4 7.4
	O	50	46	39	7	7 ...		*O*	45	45	36	6 6
HI	*I*	5940,	5004,	4212;	143,	143; 1						
(1)	*S*	49	45	41	7	7 0.6						
	O	48	48	44	5	5 ...						

Quartet System

SP	*I*	6,	4,	2						
(1)	*S*	84	68	48						
	O	89	44	67						
PP'	*I*	33.6,	4.3,	12.7;	14.4,	13.3				
(3)	*S*	48	17	13	31	30				
	O	52	20	13	28	28				
PD	*I*	120,	63,	25;	27,	32,	25;	3,	5	
(7)	*S*	50	36	23	24	26	23	8	10	
	O	50	35	23	27	28	25	7	5	
DD'	*I*	165,	83,	38,	24;	27,	34,	24		
(2)	*S*	40	28	19	15	16	18	15		
	O	43	26	14	13	19	19	14		
DF	*I*	250,	171,	112,	70;	29,	37,	28;	1.4,	2.0
(11)	*S*	47	39	31	25	16	18	16	4	4
	O	50	39	29	26	17	16	14	5	4
FF'	*I*	440,	293,	198,	154;	40,	51,	38		
(7)	*S*	40	33	27	24	12	14	12		
	O	42	35	26	23	13	14	10		
FG	*I*	420,	321,	241,	180;	29,	38,	29;	0.8,	1.0
(8)	*S*	48	42	36	31	13	14	12	2	2
	O	47	43	33	30	12	16	12	3	4
GG'	*I*	938,	679,	520,	429;	52,	68,	51;		
GH	*I*	630,	510,	410,	330;	29,	44,	29;	0.5,	0.7
HH'	*I*	1616,	1290,	1051,	896;	65,	85,	64		
(1)	*S*	42	38	34	32	9	10	8		
	O	45	26	26	34	13	12	10		

Quintet System

SP	*I*	7,	5,	3						
(6)	*S*	100	85	65						
	O	106	75	69						
PP'	*I*	112,	10,	18;	56,	54				
(2)	*S*	56	16	22	39	39				
	O	57	10	17	44	39				
PD	*I*	108,	56,	21;	28,	35,	27;	4,	9,	12
(9)	*S*	54	40	24	28	31	27	11	15	18
	O	55	40	23	32	31	26	10	16	17
DD'	*I*	180,	84,	30,	6,	0;	36,	48,	42,	24
(6)	*S*	43	29	17	8	0	19	22	20	15
	O	53	31	18	6	0	17	22	19	13

DF	*I*	220,	150	96,	56,	28;	30,	42,	40,	28;	2,	4,	4
(12)	*S*	47	39	31	24	17	17	21	20	17	5	6	6
	O	54	39	33	22	13	15	20	17	15	6	8	8
FF'	*I*	475,	304,	189,	120,	96;	53,	75,	72,	48			
(6)	*S*	40	32	25	20	18	14	16	15	13			
	O	42	31	22	17	18	14	18	16	12			
FG	*I*	456,	416,	309,	225,	162;	46,	67,	66,	45;	1.8,	3.4	3.0
(8)	*S*	47	41	35	30	26	14	17	17	14	3	4	3
	O	51	40	34	28	27	14	17	17	13	3?	3?	3?
GG'	*I*	971,	710,	520,	394,	333;	69,	101,	99,	67			
(1)	*S*	41	35	30	26	24	11	13	13	10			
	O	38	32	32	25	25	13	14	13	9			
GH	*I*	1080,	873,	699,	554,	440;	63,	92,	92,	62;	1.3,	2.4,	2.0
(1)	*S*	48	43	38	34	30	12	14	14	11	2	2	2
	O	48	36	36	24	24	...	24	24	7	..	..	...

Sextet System

SP	*I*	8,	6,	4										
(2)	*S*	116	100	83										
	O	129	100	71										
PP'	*I*	90,	6,	21,	50,	49								
PD	*I*	420,	216,	78;	120,	153,	119;	20,	50,	84				
(7)	*S*	62	45	27	33	38	33	14	21	27				
	O	70	46	25	31	35	31	15	22	24				
DD'	*I*	196,	86,	24.7,	1.07,	10.7;	44,	62,	58,	37				
(4)	*S*	47	31	17	3.5	11	22	26	26	21				
	O	46	26	16	1	9	26	27	26	20				
DF	*I*	360,	242,	149,	79.	31;	58,	89,	95,	82,	53;	2.2,	5.1,	7.2, 6.7
(4)	*S*	50	41	32	25	14	20	25	26	24	19	4	6	7 7
	O	46	41	30	25	19	23	24	25	20	18	5	8	8 8

Septet System

SP	*I*	9,	7,	5,						
(3)	*S*	133	118	99						
	O	134	116	100						
PP'	*I*	165,	8,	40;	81,	80				
PD	*I*	264,	135,	48;	81,	105,	80;	15,	40,	72
(5)	*S*	69	50	30	38	44	38	17	27	36
	O	80	52	29	36	42	32	19	25	35

DD' I	1056,	441,	105,	0,	120;	264,	375,	360,	240						
(1) *S*	53	34	17	0	18	26	32	31	25						
O	50	36	18	0	20	28	29	33	24						
DF I	390,	267,	160,	84,	31;	68,	106,	116,	101,	65;	4,	10,	18,	25,	32
(2) *S*	53	43	34	25	15	22	28	29	27	21	5	9	11	13	15
O	62	52	55	25	13	18	20	27	23	19	2	6	8	10	10

Octet System

SP	*I*	10,	8,	6						
(1)	*S*	150	134	116						
	O	133	133	133						
PP'	*I*	48,	2,	17;	31,	31				
PD	*I*	216,	110,	39;	70,	91,	69;	14,	39,	72
(1)	*S*	77	55	33	45	50	44	19	33	44
	O	85	71	28	34	57	57	17	22	28
DD'	*I*	225,	89,	18,	0.7,	38;	63,	88,	86,	57

The agreement between the observed and computed values is very close—remarkably so, considering the nature of the empirical scale on which the intensities were estimated, and the equally empirical method of evaluating this scale. It affords a comprehensive test of the theory, including as it does 231 multiplets of 46 different types, belonging to seven systems and eleven series (*SP*, *PP'*, etc.).

The only notable discordances occur when the tabular means depend on very few observed groups.

It is especially to be noted that the equations on which the computations are based are derived entirely from general principles, not a single coefficient having been adjusted empirically. It may also be remarked that King's estimates of intensity were, for the most part, made before the spectra had been analyzed and when there was no way at all to distinguish the lines of one multiplet from another. The agreement of observation and theory is therefore very striking.

Quantitative measures of intensity such as have been made for a few groups are greatly to be desired. The most important types for a critical test of the theory are the *PP'*, *PD*, *DD'* and *DF* combinations. Fortunately, excellent examples of these can be found in easily observable regions of the spectra of Ti, V, Cr, Mn and Fe, and there is good reason to hope for rapid progress.

Special attention should be given to the question whether these intensity-rules are accurately true (as they appear to be in the few groups so far measured) or only approximately so, like Landé's interval-rule.

The relation between the relative intensities of lines of the same multi-

plet, under different conditions of excitation (as in the furnace and the arc) also deserves attention. When these matters are settled, the more difficult and more important question of what determines the relative intensities of different multiplets originating in the same level will be open for investigation. It is of interest to note, however, that the expression found for the sum of the intensities of all the lines of a mulitplet, $S = 16R^2K^2(4K^2-1)$, increases rapidly with K, and for combinations originating in a given term is always greatest when $\Delta K = +1$, that is, for the combinations S–P, P–D, D–F, F–G (the first-named level being the lowest). It is well known[14] that the most persistent lines originating in any level correspond to just these transitions; for example, the diffuse series is stronger than the sharp series. It may also be worth noting that for an SS' combination, when $K = 1$, the formula would give intensity zero, and that no SS' combinations have been found in any spectrum with which the writer is acquainted.

It is hardly to be doubted that the quantum-theory, when properly applied, will clear up all these questions.

Note added June 10.—The writer received, yesterday, from Professor Sommerfeld, a copy of an article by himself and Dr. Honl (*Sitzungsber. Preuss. Akad. Wiss.*, published April 21, 1925, ix, 141 to 161) in which substantially identical formulae are reached by similar reasoning. The observational confirmation of the present paper may serve for the results of both investigations.

[1] *Zs. Physik. Braunschweig*, **15,** 1923 (189–205).

[2] *Atombau und Spektrallinien*, 4th. ed., 1924, page 589.

[3] Burger, H. C., and H. B. Dorgelo, *Zs. Physik. Braunschweig*, **23,** 1924 (258–266).

[4] Sommerfeld, Arnold, and W. Heisenberg, *Zs. Physik. Braunschweig*, **11,** 1922 (131–154).

[5] *Op. cit.*, page 589.

[6] *Zs. Physik. Braunschweig*, **15,** 1923 (189–205).

[7] *Ibid.*, **31,** 1925 (355–000).

[8] *Ibid.* (340–354).

[9] *Ibid.*, **28,** 1924 (135–141), and **29,** 1924 (241–242).

[10] See note 7.

[11] Sommerfeld, Arnold, *op. cit.*, page 654. Measures by Dorgelo at Utrecht.

[12] *Zs. Physik. Braunschweig*, **31,** 1925 (305–310), page 310.

[13] *Mt. Wilson Contr.*, Nos. 66, 76, 94, 108, 150, 181, 195, 211, 247, 274, and 283; *Astrophys. J., Chicago*, **37,** 1913 (239–281); **39,** 1914 (139–165); **41,** 1915 (86–115); **42,** 1915 (344–364); **48,** 1918 (13–34); **51,** 1920 (179–186); **52,** 1920 (232–247); **54,** 1921 (28–44); **56,** 1922 (318–339); **59,** 1924 (155–176); **60,** 1924 (282–300).

[14] Meggers, W. F., C. C. Kiess, and F. M. Walters, Jr., *J. Optical Soc. Amer.*, **9,** 1924 (355–374), page 372.

NEW LIGHT ON TWO-ELECTRON JUMPS

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Read before the Academy April 28, 1925

As is shown in preceding papers[1] the development of hot-spark spectroscopy, with the consequent opening up to precise measurement of a new spectral region, has for the first time given us a means of comparing the radiating properties of a long series of light atoms of like electronic structure such as is constituted by the seven "stripped atoms" from sodium through chlorine, viz., Na_I, Mg_{II}, Al_{III}, Si_{IV}, P_V, S_{VI}, Cl_{VII}.

A stripped atom is defined as an atom which has had all its valence electrons knocked off save the one whose jumps between the series of possible orbits give rise to the observed radiations. Stripped atoms may then be called one-valence-electron systems.

Such a series of one-valence-electron atoms as the foregoing in which the nuclear charge increases from 1 in the case of sodium to 7 in the case of chlorine has recently been shown to produce spectra which follow the X-ray spectral laws, namely the Moseley Law and the so-called "irregular doublet" law.[2]

By studying these spectra we have proved both experimentally and theoretically[3] that whenever a given type of spectral line is found to progress linearly with atomic number, i.e., whenever it is found to follow the irregular doublet law, the electron jumps giving rise to this line must all take place between orbits of the same total quantum number. *Only in this case is it possible for the observed frequencies to progress linearly with atomic number.*

By letting one more electron return to the last six of the stripped atoms mentioned above we obtain another series of atoms of like electronic structure which we have called a two-valence electron series, and similarly by letting still another electron come back to the last five of the above atoms we obtain another series of atoms of like electronic structure that we will call a three-valence-electron series.

We have recently made studies with the aid of our hot-spark spectrometry of the radiating properties of both two-valence-electron atoms and three-valence-electron atoms and have worked out the most important term values for many of them.[3] But in addition *we have found that there is one particular type of spectral flag that is flown by every one of the two-valence-electron systems with which we have worked, and that this appears to be a distinguishing, and easily recognizable characteristic of such a system.*

This flag consists of five nearly equally spaced bars or stripes the central one of which is about twice as heavy as the other four. It is seen standing

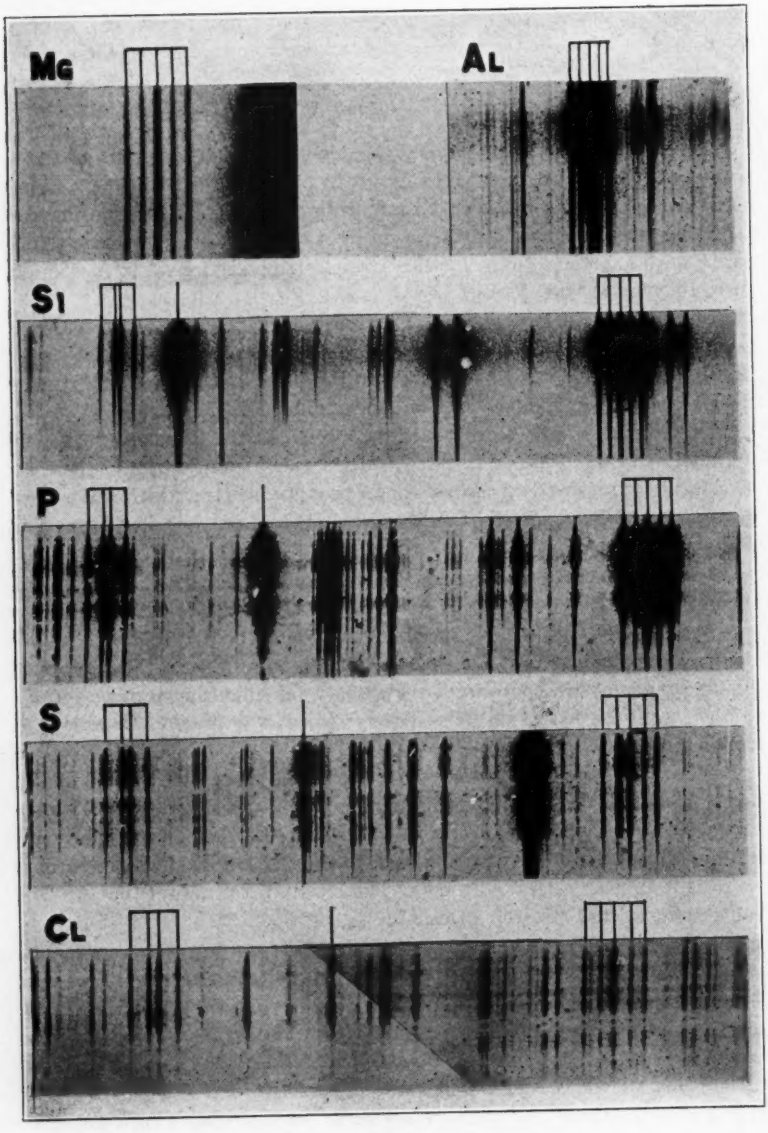

PLATE 1

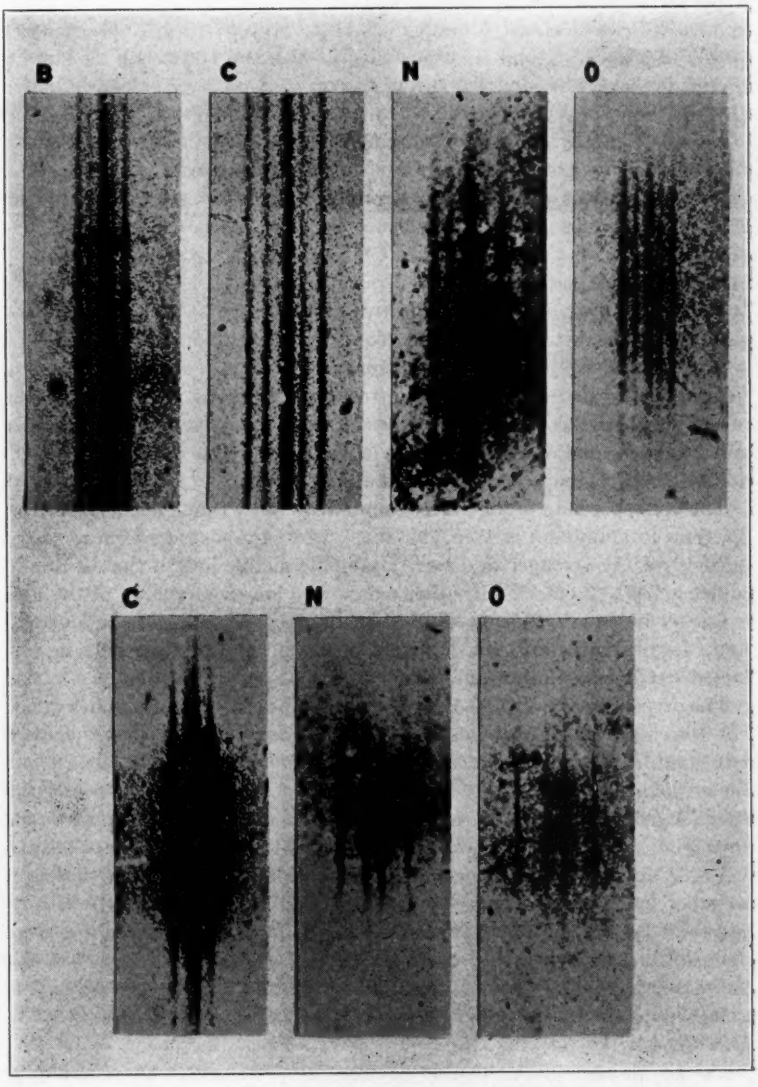

PLATE 2

out very conspicuously in two-valence-electron magnesium (Mg_I) in the upper left-hand corner of Plate 1, and on the right side of the same plate in two-valence-electron aluminum (Al_{II}), silicon (Si_{III}), phosphorus (P_{IV}), sulphur (S_V), and chlorine (Cl_{VI}). Also the upper half of Plate 2 reveals precisely the same flag in the spectra of two-valence-electron boron (B_{II}), carbon (C_{III}), nitrogen (N_{IV}), and oxygen (O_V).

Further, we have discovered that there is also a characteristic and easily recognizable flag that is flown by a three-valence-electron atom. This flag has never been observed before. It is strikingly shown along the left side of Plate 1 in the spectra of three-valence-electron silicon (Si_{II}), phosphorus (P_{III}), sulphur (S_{IV}), and chlorine (Cl_V), and in the lower half of Plate 2 in three-valence-electron carbon (C_{II}), nitrogen (N_{III}), and oxygen (O_{IV}). It consists of four bars the middle two of which are somewhat closer together than the distance from either to its outer neighbor.

Both of these flags can be definitely proved to be produced by the simultaneous jumping of two electrons, the observed frequencies being the sum of the changes in energy experienced by both of them together. Two years ago Wentzel[4] and Saunders and Russell[5] independently proved the existence of such double electron jumps in the calcium atom.

The corresponding proof is here found in the fact that the energy of the $3p_1$ term in aluminum is 48167.44 cm.$^{-1}$, while the measured energy (frequency) of the stronger and less refrangible middle line of the Al_I quadruplet is 56615.2 cm.$^{-1}$. In other words, the observed energy of the line is greater by 8448 frequency units than that corresponding to an electron jump from infinity into the p_1 level. This energy can then only be obtained from the simultaneous jumping of some other electron.

The precise nature of these two-electron jumps is here definitely revealed. *For since we find that both the quadruplet and the quintuplet groups show a very beautiful linear progression of frequency with atomic number, i.e., follow the irregular doublet law, they cannot involve any electron jump save such as takes the jumping electron or electrons between levels of the same total quantum number.* In the Be_I to O_V quintuplet series the jumps in question are definitely limited to jumps between these $2s$ and $2p$ levels, since there are no other levels of total quantum number 2. There can be little if any question then that *all of our observed two-electron jumps that produce the characteristic flags mentioned above consist in (1) a jump of one of the two electrons between a 2p and a 2s level, or a 3p and a 3s level, combined with a simultaneous jump of a second electron between two of the levels designated as* $2p_1\ 2p_2\ 2p_3$, *or* $3p_1\ 3p_2$ *and* $3p_3$.

The significance of this double electron jump is most easily seen in the case of the three-valence electron system. Such a system has normally two-valence electrons in s orbits and one in a p orbit. (See Fig. 1*A*.) When one of these s electrons absorbs energy and is pushed up to a p

orbit, the whole system of levels (orbits) about the nucleus changes because of the influence of the change in position of this electron upon the force-field about the nucleus, and the old p_1p_2 levels become displaced to the p_1^* p_2^* positions (see Fig. 1*B*). When now one of the electrons in a p^* orbit jumps back to its normal s position the other p electron has its p^* orbit knocked out from under it, so to speak, and it is obliged to jump back to one of the old p_1 p_2 positions simultaneously with the jump of the other p electron back to its s orbit. The four nearly equally intense lines of the quadruplet flag mean that in this enforced change of the electron

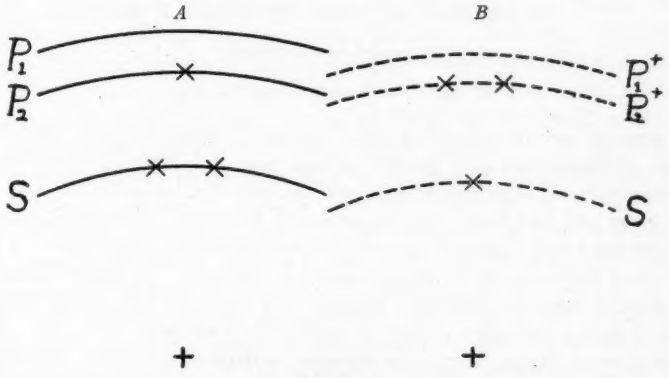

FIGURE 1

that remains in a p position it has about the same likelihood of starting from either one of the p_1^* p^* positions and landing in either one of the p_1 p_2 positions. These four nearly equal possibilities, each combined with the energy change corresponding to the p to s jump, produce the observed quadruplet. In the two-electron system there are three p levels and hence three p_2^* levels and hence six nearly equally intense lines, the two central ones overlapping and making thus the central line of double intensity characteristic of the quintuplet flag. A more detailed analysis of our results will soon appear in the *Physical Review*.

The foregoing shows very beautifully why the two-electron jumps must take place simultaneously. The fact that the energies of these two-electron jumps are added to produce a single monochromatic radiation of such frequency as is demanded by this summing-of-energy-principle raises very strikingly the question as to whether it is not forever impossible to find within an atom any single vibrating body whose frequency is that of the emitted wave. The mechanism of the transfer of energy between an atom and radiation does not seem to be of that sort. We are here in the presence of one of the most fundamental phenomena of the physical world.

[1] Bowen and Millikan, *Physic. Rev.*, **24,** 209 (1924), and **25,** 295 and **25,** 591 (1925); also *Nature*, **114,** 380 (1924).

[2] Millikan and Bowen, *Proc. Nat. Acad.*, **11,** 119 (1925).

[3] Bowen and Millikan, *Physic. Rev.*, **25,** 591 (1925).

[4] Wentzel, *Physic. Zeit.*, **24,** 106 (1923), and **25,** 182 (1924).

[5] Saunders and Russell, *Physic. Rev.*, **22,** 201 (1923), and *Astrophys. J.*, **61,** 38 (1925).

THE ANTI-STERILITY VITAMINE FAT SOLUBLE E[1]

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When rats are reared on various "synthetic" food mixtures consisting of fat, carbohydrate and protein in separate and relatively pure form together with an appropriate salt mixture and the vitamines A and B,[2] they grow well and have every appearance of health. Depending somewhat on the exact character and the proportions of the constituents of the food, they sooner or later exhibit complete sterility. In many instances a transitory period of fertility, variable in its length, follows the attainment of sexual maturity. This is usually the case with the male. But in the case of young from such females, secured either from their early fertility period or by fertility induced by certain food extracts hereinafter described, we usually observe complete sterility from the very beginning of sexual life. The sterility is a dietary deficiency disease for it can be cured or prevented by a change in dietary regime, a change involving the addition of certain single natural foods high in a new food factor, vitamine E,[3] or the addition of very much smaller amounts of extracts of those foods. The sterility disease affects males and females differently.

In the male it eventually leads to destruction of the germ cells (eventually the entire seminiferous epithelium) but this is not the case with the female, where the ovary and ovulation are unimpaired throughout life but where a highly characteristic disturbance occurs in gestation, the death and resorption of the developing young. It is necessary to insist on the peculiar character of dietary sterility thus produced in the female through lack of vitamine E, for it is only by ascertaining the existence of typical "resorption gestations" that one may be assured that he is dealing with deficiency in the specific substance E. Many other dietary delinquencies cause sterility in the female, but they all do so by interference with other steps in the reproductive mechanism than those involved in lack of E, usually by preventing oestrus, ovulation, fertilization or implantation but not by resorption after implantation has occurred. In order to establish

female sterility as due to absence of fat soluble vitamine E it is necessary to establish with certainty the existence of oestrus and ovulation, coition and implantation. Such information is best secured by use of the newer methods of studying the vaginal smear, by mating animals at the appropriate time in the oestrous cycle, by subsequent detection of the "bouchon vaginale" and residual sperm, of the cessation of cycles and finally on the 14th to 16th day, of the occurrence of erythrocytes in the smear, a positive sign of implantation. In gestations where E is low or absent, the embryos seem at first normal but sooner or later, often by the eighth day, retardation in development can be substantiated. Evident abnormality, especially monstrosity, does not occur. At some time between the twelfth and twentieth day, foetal death occurs, usually on the twelfth or thirteenth day, but for some days thereafter, the maternal part of the placenta continues to live. There may also be continued gain in the mother's weight until the twentieth or twenty-first day. This would appear to speak decisively for peculiar need on the part of the developing young for the new vitamine as against placental injury as the cause of death. Furthermore, it would appear that the maternal placenta is not altered structurally sufficiently to account for belief in impairment of its function. Subnormality is not only seen in the embryo but in the foetal parts of the placenta, both in yolk sac and allantois, especially in the former; in the yolk sac, underdevelopment of the endodermal villi and blood islands is conspicuous, whereas, in the embryo one may note impairment in the mesenchyme and its chief derivatives, the blood vessels and blood cells. The exact time of foetal death appears to vary in the case of individual mothers and what is more remarkable, in the case of some embryos as contrasted with others in the same gestation. Thus dead and living young may occupy neighboring sites in the same uterine horn. Embryos may succumb shortly after implantation, or again only shortly before term.

Large numbers of females have been reared on various "pure" food regimes and bred shortly after the sixtieth day of life. Only those exhibiting a typical resorption were now employed to trace the distribution and abundance of the new food factor E in natural foods. Shortly after the failed or resorption gestation a small amount of a single natural food stuff was now added to the ration or fed separately from it and the fate of the next gestation followed with similar care. In many instances a normal sized litter of vigorous young resulted. In others, no alteration of the sterility was secured. We have thus charted the considerable and inconsiderable possession of E on the part of common foods. It is present but never highly concentrated in a great variety of animal tissues, musculature, fat and viscera, included in the latter being pancreas, spleen, liver, heart, hypophysis and placenta. One of the most remarkable things about the content of E in animal tissues is the fact that the vitamine is low in the

viscera. It is lower in the liver than in the musculature. A daily feeding of half the total liver of rats reared on natural foods will not invoke fertility. There is failure also when the entire heart, spleen brain, kidney or testes are fed daily. The musculature and fat, on the other hand, while not a concentrated source of E, contain in their totality several times the minimum requirement for a successful gestation. E is present but extremely low in milkfat. Nine per cent of this, which is included in our basic ration, together with 15% lard, fails to prevent sterility, though with lard absent, 24% succeeds. Whole milk powder may constitute one third of the ration by weight and sterility result. Yet when whole milk powder is the sole food, its fat content, 28%, is sufficiently high to insure adequate E. There is definite evidence of a higher E content of milk when from cattle with access to fresh alfalfa pasturage. Cod liver oil, though high in vitamines A and D, is notably lacking in E. Throughout the life of animals, 9% by weight of the ration, may be constituted by cod liver oil, a single drop of which daily is adequate for A requirements and yet sterility result. In contrast with the paucity of E, even in its most abundant depots in animal tissues is its concentration in the organs of certain plants, especially in seeds and green leaves. It can be demonstrated to be unhurt after careful desiccation of such leaves (lettuce, alfalfa, pea, tea). Thus in a series of experiments, one and one-half, one and finally one-fourth gram daily of the lettuce leaf powder proved efficacious in cures. E is high in some cereals. We have found it in oats, corn and, especially, wheat, where it is low in the endosperm but concentrated in the embryo. The richness of wheat germ in E is extraordinary. We have found no other naturally desiccated substance comparable to it in value; 250 mg. daily evokes cures. In the case of both wheat germ and lettuce leaf, ether extraction of the carefully desiccated substance removes E quantitatively and secures for us oils which are efficacious in *daily*, single drop (25 mg.) administrations. E is probably present in most commercial oils so that when the latter constitute a high proportion of the diet, for instance when fed as 15% displacing lard, fertility results. Such results have been secured with Wesson oil, cocoanut oil, olive oil. Cotton seed oil when hydrogenated constitutes the substance called Crisco. As is well known, it is practically devoid of vitamine A and has hence frequently been employed instead of lard in researches where an A free diet was essential. Yet, when the fat content of our basic diet is represented by Crisco, fertility invariably results, this being in fact a curative regime. Crisco, cotton seed oil, corn oil, olive, cocoanut, walnut, peanut and flax seed oils can all be fed daily in quantities five times the required minimum of wheat germ oil without restoring fertility.

Proof of the Existence of Vitamine E in the Tissues of Animals Reared upon Natural Foods and of Its Depletion in Those Reared upon Synthetic

Diets.—We have completed a series of cannibal experiments. Sterile females reared upon "pure" food regimes were sacrificed daily and their tissues (liver, musculature and fat) fed to other females reared in an identical fashion and likewise of proven sterility. At the same time, normal females of proven fertility were similarly sacrificed and fed to other sterile "pure" food females. In all instances, the tissues of rats reared on a natural food regime were able to invoke fertility in their sterile sisters. Of even greater significance would seem the demonstration that in no instance could a cure be obtained by the administration of the same tissues from sterile females.

The Survival of Fertility in Animals Shifted from a Diet Possessing Vitamine E to One Deprived of It.—If animals are reared on a diet of natural foodstuffs and after their fertility is established, shifted to a pure food ration, they preserve their fertility for three or four months, when they lose it. Similarly, when sterile animals are cured with foods possessing the new vitamine, not only is the next gestation normal, but in some circumstances, the next two or three gestations. The survival of normal fertility is roughly dependent upon the amount of E in the curative regime. When by quantitative experiments we have determined the minimal dose of any "curative" food, i.e., one capable of immediately restoring fertility, we have been able to see the immediate loss of this fertility in the next gestation on the pure food regime.

Presence of Vitamine E in the Tissues of Normal Newborn Young.—Vitamine E is transferred from mother to offspring during intrauterine life for the tissue of new born rats acts as a cure of female dietary sterility.

Proof of the Normal Use or Wastage of Vitamine E in the Usual Metabolic Processes of the Body.—Groups of females have been reared on a natural food regime and their fertility established by trial gestations, after which they were all shifted to our standard pure diet. Half of them were bred immediately and in all instances were able to give birth to young in the next two succeeding pregnancies, the third uniformly failing. As soon as the advent of sterility was demonstrated, presumably by the exhaustion of E due to the drain of the repeated pregnancies, the other half of the animals were bred. These sisters were by this time likewise sterile. This half had been shifted to the pure food at the same time as had the first half of the group, but had been shielded from the drain of reproduction and especially placental function. It hence seems clear that the body stores of vitamine E are employed in normal metabolic processes at approximately the same rate, whether or not we have the drain of gestation.

An Excess of E Cannot Increase Fertility beyond Normal Limits.—The administration to sterile animals of foods or extracts of foods known to be twice to twenty times as rich in vitamine E as is required for the birth of living young does not increase litter size or weight, or in other ways im-

prove the performance of the reproductive mechanism beyond normal limits. This is in consonance with what we know of the action of other vitamines, there being little or no reliable evidence of advantage from an abnormally high quota of them, yet absolute need of the minimal quota and, for complete normality, for what we can call the effective quota.

Efficacy of a Single Curative Dose of Vitamine E Administered at the Beginning of Gestation.—Since the work which has previously been detailed showed a definite if transitory storage of E by the body, it seemed reasonable to suppose that a sufficiently high feeding of E on a single occasion—early in gestation—might suffice for that particular gestation. It was, in fact, found that success resulted from a single administration of the same total amount represented in twenty-two days of separate daily dosage with the minimal effective amount of wheat germ oil. The minimal effective daily dose was found to be about 25 milligrams and a single administration of 550 milligrams of the oil led in all tests to the birth of living young. Furthermore, curative foods or extracts of those foods can be fed as late as the fifth or sixth day of pregnancy and rescue the situation. Finally, the vitamine in the form of oil can be just as effectively administered parenterally—by subcutaneous or intraperitoneal injection—as by mouth.

Physical and Chemical Characteristics of the New Substance.—We come now to a consideration of the physical and chemical characteristics of the new substance. The vitamine may be called fat soluble, though its range of solubility is far greater than that of ordinary fats. While this solubility range may really be due to the solubilities of impurities as yet associated with the vitamine, it is a fact that the most concentrated fractions yet obtained have been almost completely miscible with solvents representing such a range as methyl alcohol, ethyl alcohol, ether, pentane, benzene, acetone, ethylacetate, carbon disulfide, etc. The vitamine is almost insoluble in water, yet we have repeatedly encountered its presence in watery solutions. There is enough left in the water after precipitation of calcium soaps, for instance, to be extracted with ether and effect cures. The distribution ratio between water and ether is very large, for a few extractions with an equal volume of ether effects quantitative removal. This has been established by a large number of feedings of the nonsaponifiable fraction, the residual soap always failing to produce fertility. The solubility of E in such substances as alcohol and pentane shows a large temperature coefficient and is so much greater than some of the contaminating substances—the sterols, for example—as to permit separation of the vitamine from them.

Vitamine E is remarkably stable to heat, light, air and many of the ordinary chemical reactions. As regards temperature, while the ashing of wheat germ completely destroyed the vitamine, yet heating of the germ

to 170°C so that it was greatly charred left the E unimpaired. Distillation of wheat germ oil, or a fraction out of it, in superheated steam at 180° C for several hours has not destroyed it. Distillation in vacuo up to 233°C has not in fact caused any lowering of the potency of the fractions so treated nor have any physical changes, like changes in solubility, been detected. We have not encountered evidence that daylight affects E in wheat germ oil but there would appear to be partial destruction by one hour exposure in thin layers to a powerful quartz mercury lamp. As regards oxidation, exposure of wheat germ oil for as many as twelve hours to a stream of air washed with acid and alkali and at 97°C, has not destroyed E. At normal temperatures the vitamine is remarkably stable to both acid and alkali and many chemical treatments. It dissolves unchanged for instance in saturated alcoholic hydrogen chloride. We have hydrogenated wheat germ oil in the presence of palladium at 75°C and no injury to the vitamine resulted. Further, alcoholic extracts of Crisco, a hydrogenation product of cotton seed oil, are always fairly rich in the vitamine. We have treated the germ oil with both 20% HCl and $N/10$ HCl for 20 hours at room temperatures without destruction of the vitamine. It is not destroyed by concentrated sulfuric acid. It resists the action of boiling 20% alcoholic KOH, though partial destruction would appear to occur on very prolonged hot saponification. The saponification with 20% alcoholic KOH can be carried out at 30°C without great loss of the vitamine which goes into the nonsaponifiable quota, five per cent of the oil, so that by this step alone a notable concentration of E is always attained. The nonsaponifiable quota is in turn chiefly (73%) sitosterol which is largely insoluble in pentane in the cold, an excellent solvent for vitamine E, which together with pigments and other materials can thus be washed out of the sterols, leaving them white. The sterols are inactive. The orange-red viscous oil obtained from the pentane can be treated with methyl alcohol, removing more extraneous material, the vitamine going in the alcohol portions which now can be mixed with petroleum ether and diluted to 90% MeOH, allowing an immediate separation into two layers, the petroleum ether invariably securing more of the vitamine, in fact all of it, if the distribution be done with successive fresh portions of the petroleum ether. Further purifications can now be carried out both with digitonin, boiling methyl alcohol and finally, distillations in vacuo—yet the concentration effected, of course, does not relatively compare with that effected with the first three steps of the procedure just outlined. The final yellow viscous oil does not develop crystals on long standing. It contains only a trace of ash and no nitrogen, sulphur, phosphorous or halogen. It is remarkably potent. When 5 milligrams are fed or injected under the skin of a female of proven sterility at the inception of a new gestation, normal litters of vigorous young are born and have been reared

to adulthood. Sister control rats invariably continue sterile. Furthermore the daily administration of only three tenths of a milligram of this substance throughout the life of the male results in the retention of complete normality when animals are reared and held on pure foods—a normality proven from the weight and histological condition of the testis and by weekly functional tests throughout a year, and controlled by the invariable development of sterility at the end of three months in littermate brothers held on the identical ration save for omission of the vitamine trace.

OUTLINE OF FRACTIONATION OF 6 KILOS OF WHEAT GERM

6 kilos Wheat Germ

Extracted with U.S.P. Ether in Soxhlet

Ether extract. Active. Yield: 600 grams—10%	Ether insoluble residue. Inactive.

Saponified in the cold with 20% alcoholic KOH

Nonsaponifiable matter (N.S.M.) Yield: 5%. Contains all the active material.	Soaps and Glycerol. Inactive.

Crystallized from cold pentane

Pentane-soluble red oil. Yield: 33% of the N.S.M. Contains all the active material.	Pentane insoluble solids. Sitosterol. Yield 66% of N.S.M. Inactive.

Extracted with hot MeOH

Hot methyl alcohol solution. Active.	Methyl alcohol insoluble residue. Inactive. Yield: 3% of N.S.M.

Crystallized from cold MeOH

Cold MeOH solution. Active	Precipitate from cold MeOH. Almost entirely inactive. Yield: 10% of N.S.M.

Distributed between dilute MeOH and petroleum ether.

Petroleum ether soluble. Active. Yield: 13% of N.S.M.	Dilute MeOH soluble. Inactive. Yield: 4% of N.S.M.

Sterols precipitated by digitonin.

Sterol-free orange oil. Yield: 2–3 grams. Active.

Sterols from digitonide. Inactive.

Refluxed in hot 20% alcoholic KOH

N.S.M. Contains all of the active material

Fatty acids. Inactive.

Sterols again precipitated by digitonin.

Sterol-free oil. Active.

Sterols. Inactive.

Treated with boiling MeOH

Orange solution in MeOH. Active. Yield: 700–1000 grams. (Proven active in single doses of 5–10 mgs.)

Residue insoluble in hot MeOH. Inactive. Yield: 50–100 mgs.

Distilled in vacuo

Fraction I	*Fraction II*	*Fraction III*
Up to 200° at 0.8 mm. Low activity. 38% of the total.	200–233° at 0.5 mm. Highly active. 27% of the total.	Residue above 233°. Highly active. 35% of the total.

[1] Aided by grants from the Committee for Research on Sex Problems of the National Research Council and from the Bureau of Dairying, U. S. Department of Agriculture.

[2] We have employed various proportions of casein, lard, butter and cornstarch, the vitamine B being secured from daily administration of 0.4 to 0.6 gram of whole dried yeast and A from the butter employed. In many instances, however, milkfat was omitted and the A requirements met with various levels of cod liver oil which varied from a single drop daily to two per cent by weight of the ration. The salt mixture employed was after E. V. McCollum and consisted of NaCl 0.173, $MgSO_4$ (anhyd.) 0.266, $NaH_2PO_4 + H_2O$ 0.347, K_2PO_4 0.954, $CaH_4(PO_4)_2 + H_2O$ 0.540, Fe citrate 0.118, Ca lactate 1.300.

[3] Provisionally designated in previous publications from this laboratory as vitamine X. We now so designate it on account of its serial position, following the alphabetic terminology proposed by McCollum, who names the antirachitic factor D.

THE RACIAL FACTOR IN METABOLISM

By Grace MacLeod, Elizabeth E. Crofts and Francis G. Benedict

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Communicated May 4, 1925

Popular impression has long ascribed to the Oriental, especially the Japanese, an extraordinary capacity for large amounts of work with small diets. While many dietary studies of Orientals have been made, an exact measurement of the heat production, particularly under conditions suitable for comparison with the measured heat production of other peoples, has only recently been available. Fragmentary evidence from Chinese and Japanese clinics and laboratories suggests that the basal metabolism of these people in their home environment is perceptibly lower than that of the American and European peoples. The situation is, however, considerably complicated by the fact that the climate, the diet, and particularly the nervous and muscular tension incidental to life in general is of an altogether different order among the Orientals than among those living in the rush of Western civilization. The influence of the diet and general life of Orientals and particularly the influence of the tropics upon metabolism have been the objects of considerable study and a great deal of discussion, with the decision by no means clear.

It seemed desirable, therefore, to study the metabolism under basal conditions of a group of Orientals transplanted to an American environment, so that the only variable factor would be that of race. Consequently we studied the basal metabolism of a group of 9 Oriental young women, ranging from 21 to 29 years of age, 7 Chinese and 2 Japanese, living in the environment of an American college for women, participating in all the college activities, consuming the same diet, and in every way integral factors in the college life. Due regard was given to all the standard prerequisites for basal metabolism measurements, such as complete muscular repose, an interval of 12 hours after the last meal (not too rich in protein), the absence of a febrile temperature, and the greatest possible degree of psychic repose. Observations were made on two or more days and only the averages of closely agreeing periods were used. The technique, which was in all cases that of the closed-circuit apparatus employing mouthpiece and spirometer, had been thoroughly tested, either by alcohol check-tests at the Nutrition Laboratory or by direct physiological tests, i.e., a comparison of the metabolism as measured on the same person with two entirely different types of apparatus.

The study comprised the usual anthropometric records, including in three cases direct measurements of the surface area by the Du Bois linear

formula for comparison with the surface area by the Du Bois height-weight chart. The pulse-rate, the respiration-rate, the vital capacity, and in a few cases the blood pressure were measured, but special stress was laid upon the determination of the basal metabolism, i.e., the oxygen consumption. In one case a study of the skin temperature was made.

The findings may be summarized as follows. The vital capacity was, judged by American standards, very low, that is, 14.3 c.c. per centimeter of height and 1.54 liters per square meter of surface area, as compared with the normal standards for women of not far from 20.0 c.c. and 2.0 liters, respectively. The blood pressure of the three subjects measured was normal.

The average pulse-rate for the entire group was 60 beats per minute, with a minimum of 54 and a maximum of 64, i.e., values that are at the lower, if not indeed slightly below the lower limits for normal American women. The average respiration-rate of 14 was not unusual.

As an index of the general state of nutrition the pelidisi was computed, and was found to range from 88 to 99, averaging 92; i.e., there was a slight tendency for the pelidisi to be a little lower than the assumed normal of about 95.

The most significant findings had to deal with the basal metabolism, which in all cases was below the accepted normal and in most cases strikingly low, the average metabolism being 10.4 per cent below the Harris and Benedict prediction standards for women. As is to be expected, comparisons with the Aub and Du Bois standard (i.e., the heat per square meter of body surface area) and with the Dreyer standard gave essentially the same picture of a low metabolism.

There was nothing in the physiological measurements of these young women to indicate abnormality, and we believe that our data show that transplanted Orientals retain their inherited low metabolism.

Evidence secured in our and in other laboratories on American young women indicates that the present standards are probably about 5 per cent too high and that a part of the seeming discrepancy between Oriental and American women may be ascribed to faulty standards. But even if the standards for American women are corrected by 5 per cent, the average metabolism of our 9 Orientals would still be below the corrected standards. This fact strongly suggests a specifically low metabolism of the Oriental woman.

The standards for basal metabolism have already by common consent been modified for the effects of sex and age. It is now suggested that the correction for difference in sex be increased from 7 per cent (the correction made at present) to 12 per cent, and that a racial effect in the direction of a lower metabolism be recognized, at least with Chinese and Japanese women.

The detailed figures are about to be printed in the *American Journal of Physiology*.

PROTOPLASMIC ACTION OF COPPER AND GOLD

By Carl Voegtlin, J. M. Johnson and Helen A. Dyer

Hygienic Laboratory, U. S. Public Health Service

Read before the Academy April 27, 1925

The main object of this work was to throw some light on the intimate mechanism whereby copper and gold salts exert their toxic action on protoplasm. These salts are highly toxic to all sorts of animal and vegetable cells. Naegeli found that the minute traces of copper, which are dissolved by water exposed to metallic copper, are sufficient to kill spirogyra. He introduced the term oligodynamic action in order to distinguish the action of these dilute solutions from the action of more concentrated ones. Ringer and Locke made similar observations on tadpoles and on the isolated heart. More recently, the subject has aroused renewed interest in connection with the antiseptic action of the heavy metals and the production of a disease in animals kept on food containing copper, which appears to be identical with hemochromatosis in man. The subject is of further interest on account of the occurrence of traces of copper in human tissues under apparently normal conditions.

Practically no knowledge exists as to the mechanism of this toxic action. Naegeli and Saxl in particular have surrounded the oligodynamic action with a great deal of mystery in assuming that the absolute amount of heavy metal involved is so small as to exclude a chemical effect and hence they assume that this action is due to a physical force of unknown nature. They neglect to take into consideration the relative mass of protoplasm and of heavy metal salt. A thread of spirogyra may weigh only a fraction of a milligram and on exposure to a large volume (100 cc.) of an even very dilute copper sulphate solution (1:10 million) enough metal ions are present to exert a toxic effect by chemical means. The large surface of such minute organisms as spirogyra and bacteria furnishes ideal conditions for adsorption of the metal with subsequent storage in the cells. Moreover, it was found that in a given volume of a toxic solution the toxic effect disappears when the absolute amount of spirogyra is increased. We confirmed this observation on tadpoles. Two small tadpoles, each weighing about 10 mg., were killed within $2^1/_2$ hours when placed into 200 cc. of a solution of copper sulphate 1:1 million, whereas two larger tadpoles weighing 500 mg. survived in a solution of 1:500,000. The fact that the larger tadpoles survived is simply due to the larger mass of protoplasm and hence the toxic threshold concentration could not be reached. All the evidence speaks in favor of a chemical action. What are then the chemical constituents of protoplasm which enter into reaction with these traces of heavy metals and how is it that the cell is injured and killed? The action of higher

concentrations of heavy metals salts may be due to the coagulating action on cellular proteins, but in high dilutions it would be difficult to conceive how such small amounts of metal ions by combining with a relatively small fraction of protein molecules present could bring about death. Is it not more likely that in this case the metal ions exert their action on a vital constituent which is present only in small amounts? Glutathione, a dipeptide of glutamic acid and cysteine, is present in mammalian tissues, yeast, probably also in other organisms, and is concerned in cellular oxidation-reduction phenomena. We found that glutathione in its reduced (R.SH) form reacts with copper sulphate and gold chloride with the formation of copper and gold derivatives, respectively. Similar compounds were obtained from cysteine. A part of the SH group appears to be oxidized by the metal salt with the conversion of the metal ion to the cuprous and aurous state, respectively. If these reactions also occur in living cells, then the toxic action might be due essentially to a disturbance of the glutathione equilibrium

$$\begin{matrix} \text{R.SH} \\ \\ \text{R.SH} \end{matrix} \rightleftharpoons \begin{matrix} \text{R.S} \\ \\ \text{R.S} \end{matrix} + \text{H}_2$$

and death might be conceived as a special type of asphyxia. It, therefore, should be possible to protect protoplasm against metal ions by furnishing the former with an extra supply of reduced glutathione, or possibly some other SH compounds.

It was found that albino rats survive from an intravenous injection of a minimum lethal dose of sodium cupritartrate, if injected also with a sufficient amount of glutathione or cysteine (10 mol glutathione per 1 mol Cu). None of the other amino acids, nor glucose, lecithin and other protoplasmic constituents tested have any protective action. The hemolysis and loss of body weight of sublethal doses of copper can also be prevented by glutathione. Sodium gold chloride is also detoxified by reduced glutathione and incompletely by cysteine, but not at all by the other substances.

The oligodynamic action of $CuSO_4$ on spirogyra is also antagonized under certain conditions by the presence of glutathione but not by an equimolecular concentration of the other compounds. A few similar observations were also made on tadpoles. So far the observations appear to fully substantiate the above-mentioned hypothesis of the toxic action of copper and gold on protoplasm from widely different sources. These results should be corroborated on other organisms, especially bacteria, before a generalization could be accepted.

ACTIVITIES OF COLONIAL ANIMALS. III. THE INTERRELATION OF ZOÖIDS IN SOFT CORALS

By G. H. Parker

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Communicated May 12, 1925

While I was resident at the Barro Colorado Island Laboratory in Gatun Lake, Panama Canal, I had the opportunity of studying the marine faunas of the Atlantic and of the Pacific sides of the isthmus. On the Pacific side there are extensive flats off the city of Panama and in the sea pools exposed on these flats at very low tides a number of living gorgonians may be seen. Three of these were in fair abundance and have been kindly identified for me by Dr. Henry B. Bigelow as Leptogorgia alba (Duch. and Mich.), L. ramulus (Val.) and Muricea hispida Verrill (see Verrill, 1868–'70, pp. 398, 394, 422).

All three are favorable for tests on the interrelation of their zoöids. Leptogorgia alba is composed of long, infrequently branched, white stems about a millimeter in diameter and with many small scattered zoöids. L. ramulus spreads out in one plane for the most part; its abundant branches, which are covered with many zoöids, often anastomose and may be two or more millimeters in width. Muricea hispida is a thick-stemmed, coarsely branching species with numerous scattered zoöids.

When these forms were first examined in their native pools their zoöids were found to be fully and freely expanded. On breaking them off from their rocky foundations and transferring them to small dishes of seawater, their zoöids at first contracted and then quickly re-expanded and remained so in their new situations in the dishes. This was very favorable for tests.

If with a needle a mechanical stimulus is applied to a spot on a given branch of any one of the three species named, the zoöids in immediate proximity to the spot stimulated contract while those on the rest of the branch remain uninfluenced. The response is as local as one could well imagine. Even if a branch is cut off while the colony is under seawater, only the zoöids immediately about the region of the cut contract and these quickly re-expand. No wave of contraction among the zoöids passes up or down the parts of the cut branch. In this respect the independence of the separate zoöids seems to be most striking and is in strong contrast with the condition seen in Renilla (Parker, 1920) where upon stimulation, such as is produced by cutting the flesh of the colony, not only do the zoöids retract locally but a general wave of zoöidal response spreads over the whole colony.

The tests thus far described were carried out during the first few visits to the flats. On a subsequent trip the zoöids on all three species of gor-

gonians were found to be completely withdrawn. A comparison of the conditions showed that the earlier inspections had been made with an overcast sky and the later one under full sunlight. When the colonies with contracted zoöids in full sunlight were broken from their natural attachments and put in dishes of fresh seawater in the shadow of a rock, the zoöids soon emerged and behaved otherwise as they had on the overcast days. If, now, such a colony, with its zoöids expanded in the shade, is illuminated by reflecting on it with a hand mirror a small circumscribed beam of sunlight, the zoöids of that part of the colony thus illuminated contract quickly and remain so as long as the light is on them whereas in the unilluminated portion the zoöids continue in the expanded state. The line of separation between the expanded and the contracted zoöids corresponds so closely to the limits of the beam of light that there is no reason to suppose that the disturbance set up by the light is in any way transmitted beyond the region of illumination.

The mutual reactions of the zoöids in the three species of gorgonians mentioned in this paper are in striking contrast with the reactions of the zoöids in such a colony as Renilla in that the gorgonian zoöids seem to be entirely independent of one another notwithstanding the fact that they are imbedded in a common flesh whereas the zoöids in Renilla are mutually responsive as a whole. In the gorgonians mechanical or photic stimulation of a part of the colony will induce local zoöidal contraction without, however, leading to any response on the part of the neighboring zoöids. The zoöids in Renilla on the other hand are held together by connections probably in the nature of a nerve-net which seems to be quite lacking in the gorgonian colonies. In the paper on Renilla already mentioned I have stated that the activities of this animal are colonial in scope rather than zoöidal and that the zoöid as a unit is dominated by the colony. In the gorgonians tested the reverse appears to be the case; the colony is essentially zoöidal in that each zoöid is in a striking way essentially independent of its neighbors. Whether this distinction is a general one for gorgonians and pennatulids cannot, of course, be stated because of the very limited range of material tested, but it is one worthy of further consideration.

Parker, G. H. 1920. Activities of Colonial Animals. II. Neuromuscular Movements and Phosphorescence of Renilla. *J. Exper. Zoöl.*, **31**, pp. 475–514.

Verrill, A. E. 1868–'70. Review of the Corals and Polyps of the West Coast of America. *Trans. Connecticut Acad.*, **1**, pp. 377–558.

PERIODICITY OF REPRODUCTION, INFECTION AND RESISTANCE IN BIRD MALARIA

By Lucy Graves Taliaferro

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Communicated April 23, 1925

An analysis of the effects of resistance has been carried out on bird malaria (*Plasmodium praecox* from the English sparrow inoculated into canaries) in a manner similar to the investigations which the present writer carried out in collaboration with W. H. Taliaferro (*Amer. J. Hyg.*, **2**, 1922, 264–319. See also Taliaferro, W. H., *J. Exp. Med.*, **39**, 1924, 171–190 and Coventry, F. A., *Amer. J. Hyg.*, **5**, 1925, 127–144). In this work resistance has been used to denote collectively those factors, either active or passive, which operate adversely against the parasite. Thus, it is easy to see that the number of parasites would increase at a uniform rate if no resistance developed, whereas, if they do not so increase, some type of resistance may be operative. Any effect on the number of parasites, however, may be brought about by one or both of two factors; first, the rate of reproduction of the parasites may be retarded or inhibited, or second, the organisms after they are formed may be destroyed, as the following equation shows:

(1) *Parasites at any time* = (2) *Number produced by reproduction* (1st factor) − (3) *Number destroyed* (2nd factor).

If we can obtain any two of these terms, we can evaluate the third and arrive at a fairly accurate conclusion with regard to which factors in resistance are operative.

In all of this work, the first and second members of the equation have been ascertained and the third evaluated. Thus, the first term was easily obtained by making frequent number counts during the course of an infection. The second member, in the trypanosome work, was obtained by comparing the variability in size of different samples of organisms taken at stated intervals throughout an infection. Such comparisons of variability in size may be used to measure the rate of reproduction of the parasites, because whenever reproduction is going on, there will be a great variability in size, whereas when there is none, the variability will be low. As a measure of this variability, the coefficient of variation for total length was used. The advantage of this measure is that it is independent of the second effect of resistance.

This method of obtaining a measure of the second term of the equation was not available for bird malaria because if reproduction is going on in the

trypanosome infections, all stages of growth and division may be found on a blood smear, whereas in bird malaria, on account of the *periodic cycle* of growth and reproduction, only one particular growth stage may be found at any one time (see fig. 1). Another measure of the rate of reproduction was therefore developed which works out as follows. At the time when large schizonts (asexual stages which later undergo sporulation) are in the blood, the mean size of the parasites is high; as sporulation takes

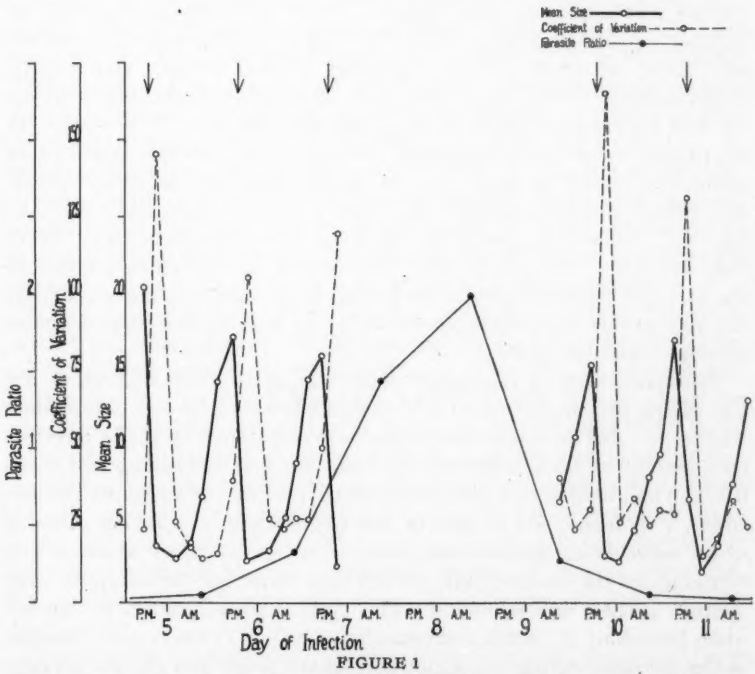


FIGURE 1

Major portion of acute and chronic infection in Bird 61. Number of parasites per 10 red blood cells, and mean size and coefficient of variation for asexual stages during two series of observations. The distance between peaks of the mean is a measure of the rate of reproduction of the parasites which is independent of the number of parasites destroyed.

place, the mean size will immediately drop, after which it will gradually rise as growth ensues until the next period of sporulation. This goes on indefinitely. The length of time between the high values of mean size is actually a measure of the time it takes a merozoite (small form) to become a full-grown schizont (asexual form ready to sporulate), and hence will give us the rate of reproduction of the parasites. Such a measure has the

same advantage that the measure devised for the trypanosome had in that it is independent of any factor which might destroy large numbers of organisms.

The Sargents (*Ann. Inst. Pasteur*, **32**, 1918, 382–388), Ben Harel (*Amer. J. Hyg.*, **3**, 1923, 652–685), and Boyd (MS.) have studied the course of an infection in bird malaria. They found by making daily parasite counts that the typical infection runs somewhat as follows. The parasites, after an incubation period, increase in numbers in the blood and sometimes become very numerous (*acute period*). If the bird does not die, they invariably reach a peak from which they rapidly decrease (*crisis*) and are only found in small numbers thereafter (*chronic period*). Eventually, they disappear (*latent period*) although under unfavorable conditions for the bird they may reappear in small numbers (*relapse*). A number curve for part of the acute period, crisis, and part of the chronic period of an infection is shown in figure 1. The rest of the typical infection (latent and relapse periods), which may cover a year or more, has had to be omitted. Suffice it to say that during these periods the parasites are never very numerous. From this work, it is evident that a sudden decrease of the parasites terminates the acute period of the infection, and as they do not migrate into the tissues, the second factor in resistance (destruction of parasites) must be operative.

The main object of the present work was to ascertain whether or not this destruction was associated with any inhibition of the *rate* of reproduction *per se*. By making several series* of blood smears at 4 hour intervals for a number of birds throughout the infection, and by drawing and measuring 50 parasites on each slide, it was discovered that in a recently isolated strain, a 24 hour cycle of growth and reproduction was taking place *no matter when the observations were made.* The sexual stages which do not reproduce in the bird, but are the infective forms for the mosquito were omitted in these measurements. This cycle is clearly shown in figure 1 (data from Bird 61) which represents graphically (1) the number changes of the parasites during the major part of the acute and chronic periods, (2) two series of data showing the changes in mean size and (3) the coefficient of variation for the same periods. The number curve is typical, as described above. The mean curve, whenever data was drawn up, is high around 5:00 P.M., falls pretty sharply at 9:00 P.M. and gradually rises through the rest of the night and following day. The coefficient of variation for the mean size also shows the cyclic nature of the infections. It is approximately constant except at the time of sporulation. This is due to the fact that growth takes place fairly uniformly—hence, the coefficient of variation, is, generally speaking, constant—but sporulation causes a marked variability in size of the forms encountered (both very large and very small forms occur)—hence, the coefficient of variation will be corre-

spondingly high. The arrows in the figure show the time of sporulation. Detailed studies have been made on 6 birds, and additional observations on 11 others. (In one strain the cycle was of 30 hours duration.) From these data, we may conclude that the reproduction of the parasites progressed at a uniform rate during the acute, crisis, chronic and relapse periods—in fact whenever the parasites were numerous enough to get a statistically valid sample. Hence, in terms of resistance, the host did not develop any resistance directed toward a retardation or inhibition of reproduction.

Finally, this study seems to have a direct bearing on the problem of relapse. Among the many hypotheses advanced (see Hegner and Taliaferro, W. H., *Human Protozoölogy*, 1924, pp. 349–351), Bignami, Ross, Whitmore and others believe that the asexual cycle continues throughout the latent period although the parasites are too few to find until some condition adverse to the host causes a relapse. This view is borne out by the present study. I found that not only were the parasites reproducing at the same rate whenever found, but various parts of the cycle occurred at exactly the same time after as before the latent period. It seems safe to suppose then that even when the parasites are too few to find, they are continuing their cycle of growth and sporulation uninterruptedly.

In the various periods of the infection outlined previously, the acute part of the infection undoubtedly presents the optimum conditions for the life of the parasite. Consequently, it is interesting to ascertain just what proportion of the parasites survive during this period. Since the reproduction rate is constant for a given strain, and since microscopic examination indicates that, on the average, each schizont produces 15.5 merozoites, we can compare the number of merozoites actually produced with the number which must survive to give us the observed increases of the parasites in the blood. The results indicate that the majority of merozoites produced at each sporulation period perish, and conversely, that only a small but fairly constant proportion survive and complete their development.

Our tentative conclusion in regard to an infection with bird malaria may be briefly summarized as follows: The asexual stages of the parasite are found in the peripheral blood of an infected bird in varying numbers and undergo their cycle of development and reproduction at the same rate throughout the entire course of the infection. From the beginning the bird seems to possess a natural resistance because only a part of the merozoites which are produced have been found to survive. At or near the crisis, the bird develops a resistance directed toward a destruction of the parasites which becomes more potent as time goes on although it may be temporarily minimized. As a consequence, at the crisis, a large number of parasites are destroyed; during the chronic period, the number destroyed equals or slightly exceeds the number produced by reproduction; during

the latent period, the destruction of the parasites keeps the number in the blood below the number at which they may be found by microscopical examination (they are still present in small numbers as shown by Whitmore, *Johns Hopkins Hosp. Bull.*, **29**, 1918, 62–67); and during the relapse, the destruction is temporarily lessened so they accumulate in the blood. At no time is there evidence of any retardation of the rate of reproduction as found by Taliaferro, W. H. (1924, *loc. cit.*) in *Trypanosoma lewisi*.

* A bird has such a relatively small amount of blood that it is not possible to make blood smears at four hour intervals throughout a single infection. Therefore, several series of slides were made from each bird allowing several days for recuperation. From the birds as a whole, however, data was obtained covering every period of the infection except the latent period, of course, when no parasites can be found.

THE CHROMOSOMES OF ZEA MAYS

By Emma L. Fisk

University of Wisconsin

Communicated April 22, 1925

Zea Mays L. offers an interesting field for the study of chromosome behavior, particularly in view of the importance the plant has assumed in problems of genetics.

Determinations of the chromosome number in several varieties of maize have been made by Kuwada. In 1911, he[1] reported that the haploid number varies from 9 to 12 in different races, at the same time expressing his view that 12 is to be considered as the original number. In 1919, Kuwada[2] concluded that, with the exception of the "sugar corns," the chromosome numbers are 10 (haploid) and 20 (diploid) and that the plants most closely related to Zea as possible ancestral species have a somatic number of 20. The sugar corns, according to Kuwada, show considerable variation, from 20 to 25 somatic chromosomes being present. In the majority of plants of the sugar-corn type, the haploid number was found to be 12.

Longley,[3] in a study of maize and maize relatives in 1924, reported that, in the four varieties of corn studied, his investigations failed to show any deviation from 10 as the haploid chromosome number. Hybrids between *Zea Mays* and *Euchlaena mexicana* show 10 bivalents in the heterotypic division. The number of chromosome elements in a cross between *Zea Mays* and *Euchlaena perennis* is 30, and their distribution at meiosis is very irregular.

Since the writing of this paper, an article has appeared by Kiesselbach and Petersen[4] giving the results of chromosome counts, by the aceto-carmine method, of a number of commercial and inbred strains of dent maize,

and of several other varieties including Chinese maize, Country Gentleman, and Red Flower. All the forms studied were found to have ten as the haploid number of chromosomes.

The present writer has had in progress for the past five years a cytological study of about twenty-five pedigreed strains of corn, from material obtained from Professor E. W. Lindstrom. Investigations include such varieties as yellow flint, red flint, Golden Glow yellow dent, floury, dwarf starchy, Golden, "anther-eared" semi-dwarf, pod corn, and *Zea ramosa*. Among the sugar corns investigated were selfed strains of Golden Bantam, Evergreen, Crosby, and Black Mexican sweet corn. F_1 crosses between yellow dent and Crosby sweet corn, between dent and Black Mexican sweet, and between yellow flint and Crosby sweet corn are included in the study. With paraffin material, sections of sufficient thickness were used so that they showed a complete nucleus, or the cytoplasm above and below a chromosome group. All counts were made from camera lucida drawings.

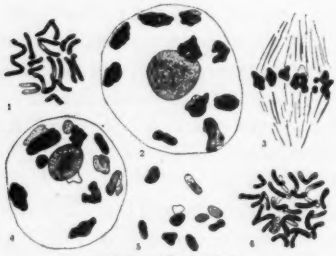
In general, a haploid number of 10 chromosomes, and a corresponding diploid or somatic number of 20, have been found. Cells of root tips in division in most forms studied give a definite number of 20. Occasionally one with only 19 is found, but in several cases the unusual length of one chromosome indicated that segmentation was not completed, or that two chromosomes might be end to end or overlapping. In a few instances, it was a question between 20 and 21, but no clear cases over 20 were found, with the exception of Black Mexican sweet corn, which will be considered later.

There is a distinct variation in the length of the somatic chromosomes, although I do not find it possible to separate them into two distinct classes of long and short chromosomes, such as were suggested by Kuwada. Although there is a distinct variation in cell size in different parts of a root tip, the chromosome mass remains fairly constant. Cells from dwarf plants of the floury variety failed to show any difference in chromosome number or bulk from those from normal-sized plants. As to the individuality of the chromosomes within a given sub-species, while the differences between the chromosomes are not clearly marked, the number of chromosomes of respective lengths is found to be markedly constant. Two curved chromosomes of medium length were particularly noticeable in many of the sections of all varieties studied. Often one or two chromosomes longer than the others stood out clearly, especially in the flint and floury varieties, and in *Zea ramosa*. Some evidence was found of the pairing of somatic chromosomes. In some preparations, nine or ten distinct pairs are suggested, as in figure 1. In other cells only a few pairs may be detected. In rare instances, a chromosome was apparently "cross-segmenting."

According to most investigators, stages in microsporogenesis offer much

more favorable material than somatic divisions for chromosome counts and for a study of recognizable differences in size and shape of chromosomes. Even in these stages, in Zea, however, the individuality of the various chromosomes is not as marked as in the cases of some genera described by Sakamura, Taylor, and others. Several hundreds of counts have been made, in diakinesis, on the heterotypic and homoeotypic equatorial plates, in the anaphases, and a few in interkinesis. With several exceptions to be noted later, the haploid number is distinctly 10.

In diakinesis, the bivalent nature of the chromosomes is clearly evident (fig. 2). In early diakinesis, they may assume rather grotesque shapes,



FIGURES 1 TO 6

Fig. 1. Polar view of equatorial plate, showing pairing of somatic chromosomes in root tip. Golden Bantam sweet corn. Fig. 2. Pollen mother cell at diakinesis, showing 10 bivalents. Crosby sweet corn. Fig. 3. Equatorial plate of heterotypic division. Red flint corn. Fig. 4. Diakinesis with 8–10 bivalents and extra bodies. Black Mexican sweet corn. Fig. 5. Polar view of equatorial plate, heterotypic division. 10 gemeni. Evergreen sweet corn. Fig. 6. Equatorial plate, from cellular endosperm tissue, showing 30 chromosomes. Evergreen sweet corn.

FIGURE 7

Bivalents from five cells in diakinesis from one plant, arranged according to apparent size and shape. Golden Glow Yellow Dent crossed by Crosby sweet corn.

being very ragged as to general appearance. Apparently the condensation of the gemini does not take place with equal rapidity in all pairs of chromosomes, for, in some cells, certain ring-shaped bivalents are noticed whereas the other pairs remain as irregular twisted masses. As condensation proceeds further, the gemini take on the shapes in which their specific identity is most marked. When they lie with their long axes perpendicular to the line of vision, at least four and probably six or more distinct rings are seen, as in figure 2. The impossibility of obtaining all the gemini in flat view in one cell renders an absolute determination out of the question. Slight differences are evident in the sizes and shapes of these rings, certain ones appearing practically closed, others with halves barely joined, and still others with ends overlapping. In some varieties there appears a greater tendency toward the forms of Y's and X's. Occasionally a wide V suggests a tendency toward separation of the univalents from each other. A constantly recognizable bivalent in certain forms is a large one in

which the twisted nature of the pair is evident. Figure 7 represents the ten gemini seen in the heterotypic diakinesis in each of five pollen mother cells, arranged according to their apparent size and shape. The writer wishes to emphasize that such an attempt must be partially inaccurate, because some gemini may be unfavorably placed in relation to the line of vision, or because some may be partially covered by the nucleole. It is in late diakinesis that determination of chromosome numbers is most accurately made.

After the nuclear membrane has disappeared and the multipolar spindle is formed, the chromosomes are so condensed that their specific nature is lost. There are slight variations in the size of chromosome pairs, as in figure 5, and occasionally a ring may still be indicated.

On the equatorial plate of the heterotypic division the paired chromosomes are so regularly arranged that they may best be counted in polar view. One pair of small rounded chromosomes, which apparently separate slightly earlier than the others, is often visible, as in figure 3. Lobed or V- or Y-shaped univalents are seen in anaphase stages and at interkinesis, indicating a preparation for the second division. Occasional counts may be made in interkinesis. Homoeotypic equatorial plates furnish favorable material for determining actual numbers present. A slight variation in size of chromosomes is noted.

An apparent exception to the general rule of $X = 10$ and $2X = 20$ is found in Black Mexican sweet corn, from material selfed for one year. In one adventitious root tip, a variation from 20 to 23 chromosomes is found, the number in the majority of instances, however, being 22. In diakinesis, in material from several plants studied, there seem to be present from 8 to 11 chromosome pairs, with extra rounded or elongated bodies varying from 0 to 4 in number, as well as in size and shape. Occasionally they appear as 11 bivalents, but more often as a smaller number of bivalents with other rounded bodies or longer black-staining bodies which might be univalents, as illustrated by figure 4. From present results it is impossible to state whether all these bodies are chromatin in nature or whether some are bodies derived from the nucleole. Their presence renders the actual chromosome count very uncertain. Certain equatorial-plate stages and anaphases indicate 10 chromosome pairs only; others show the presence of a pair of rounded bodies on the spindle on either side of and away from the equator where the chromosomes are arranged; still others show one such body either on the spindle or in the cytoplasm. The question as to whether or not these extra bodies behave regularly on the spindle is as yet unanswered.

The presence of such dark-staining bodies on the spindle apart from the chromosome mass or in the cytoplasm is not limited to Black Mexican material; occasional instances have been found in Golden Bantam sweet corn, and one case in flint corn, this time outside the nuclear membrane.

Even in somatic divisions in corn, the nucleole often persists, after the nuclear membrane has disappeared, as a small rounded body which might be mistaken for a chromosome in end view. Budding of the nucleole is not infrequent. In the heterotypic division, it is often found on the multipolar spindle as a body slightly larger than the gemini. As it diminishes in size, it may reach a stage at which it is practically impossible to distinguish it from the bivalents. From time to time, it is noted that not all the chromatic material in a nucleus has gone into the formation of the chromosomes; delicate strands may connect several chromosomes in late diakinesis. Instances are found of a mass of small dark-staining bodies of varying size in the nucleus, in preparations in which most cells appear normal, suggesting material from the nucleole.

Several instances of non-disjunction during the heterotypic division have been found in Golden Bantam sweet corn, there being 9 chromosomes in one daughter cell at interkinesis or on the homoeotypic spindle and 11 in the other daughter cell.

In preliminary cytological studies on the endosperm of a number of varieties of corn, no cases of irregularities in division figures have been found. In cellular endosperm, in plants of Evergreen and Black Mexican sweet corn, chromosome counts were made from prophases and from equatorial plates (fig. 6). Results indicate clearly a $3X$ condition, with 30 as the definite chromosome number in the endosperm.

[1] Kuwada, Y., *Bot. Mag., Tokyo*, **25**, 1911 (163–181).
[2] Kuwada, Y., *J. Coll. Sci., Tokyo*, **39**, 1919 (1–148).
[3] Longley, A. E., *J. Agric. Research*, **28**, 1924 (673–682).
[4] Kiesselbach, T. A., and Petersen, N. F., *Genetics*, **10**, 1925 (80–85).

MEASUREMENT OF THE SPECTROSCOPIC BINARY STAR MIZAR WITH THE INTERFEROMETER

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The second-magnitude star Mizar (Zeta Ursae Majoris) is particularly interesting on account of its history. Together with Alcor it forms a visual binary known from the earliest times. It was the first star known to have a telescopic companion, the discovery being made by Riccioli in 1650. The telescopic pair was the first double to be measured photographically, by Bond in 1857, and one of the components was the first star to be announced as a spectroscopic binary, in 1889 by E. C. Pickering.

Elementary calculations indicated that the spectroscopic pair was likely to be just within the resolving power of the 20-foot Michelson interferometer attached to the 100-inch Hooker telescope. The interferometer is not well adapted to the measurement of double stars, since it is necessary to lift the united beam and Cassegrain "cage" with the crane in order to rotate for position angle, and since further the "cage" has not been accurately fitted to the tube for positions other than those used in ordinary observing. Considerable time is therefore necessary to find the fringes at each setting.

On the nights of April 15 to 19 inclusive, a series of preliminary measures was made. The results are listed in the following table. The settings were made in position angles N–S, NE–SE, E–W, SE–NW, and the angles given are those indicated by the relative values of the visibility for the position angles stated. The interferometer seeing was fair, although the ordinary seeing was poor, 1 to <1 on a scale of 10. Every setting of the instrument was checked on one of the three stars β, ϵ or η Ursae Majoris. The value of the separation of the components of a double star is found from the formula $\alpha'' = 103{,}133\ \lambda/D$, where λ is taken as 5.4×10^{-5} cm. for an A type star and D is the separation of the mirrors in cm.

Greenwich Civil Time 1925	Position Angle		Separation of Mirrors	Separation of Stars	
	Measured	Calculated	Feet	Measured	Calculated
April 16.3	247°	251°	16	0".013	0".014
17.3	260°	259°	16	0.013	0.014
19.3	280°	279°	18	0.011	0.012
20.3	292°	298°	18	0.011	0.009
Change	45°	47°		0.002	0.005

Although these observations by themselves are insufficient for the determination of an orbit, comparison with the well determined spectroscopic data is of interest. These give all the elements of the orbit except the node and inclination, which may be provisionally determined from two observed position angles. Since the parallax of the star is very well known, the separation of the components can be computed and the agreement with observation affords a valuable check. Using Hadleys' spectroscopic elements $P = 20^d.536$, $\omega = 104°$, $e = 0.53$, $(a_1 + a_2) \sin i = 32{,}880{,}000$ km., a periastron passage is found to occur on April 22.2 G.C.T. A plot of the observations indicates that $\Omega = 285°$, $\gamma = 50°$. With these values and the parallax 0".045 derived from the group motion, the computed positions are as given in the table. Although they are highly provisional, the agreement with observation perhaps justifies their insertion.

I am indebted to Dr. Russell for suggesting that the star might be resolvable and for the provisional calculations.

DISTRIBUTION OF THE STARS WITH RESPECT TO BRIGHTNESS AND DISTANCE FROM THE MILKY WAY

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The stars whose parallaxes can be measured, either by trigonometric or spectroscopic methods, are all so near that knowledge of their distances gives no clue to the form and size of the stellar system. The structural features of the system must be learned from other data, foremost among which are counts of stars of different magnitudes in different parts of the sky.

Were the stars all of the same intrinsic brightness, the determination of their distribution in space would be simple. The obvious differences in apparent brightness would then be merely the effect of differences in distance, which could easily be calculated with the aid of the inverse square law. Star-counts, together with the known distance of a single star, would thus determine the distribution of all objects within reach of our telescopes. Actually, the stars are not of the same intrinsic brightness; in fact, the range in stellar luminosity is enormous, at least 20 magnitudes, corresponding to a ratio in light-intensity of 100,000,000 to 1. This seriously complicates the problem, but does not render it insoluble. Allowance for differences in luminosity can be made, provided the numbers of stars of different intrinsic brightness in a given volume of space—the luminosity law—can be learned.

The determination of the luminosity law is in itself an extremely troublesome matter. We have an excellent approximation for the relative numbers of stars intrinsically brighter than the sun, which stands about midway in the known range of luminosity, and a general indication of the frequencies of fainter stars. A more exact specification of the law is by no means hopeless and will at once clear the way for a full utilization of star-counts, whose value is thus in no wise impaired by the complexities of the problem.

The importance of data on the distribution of the stars over the sky has been recognized since the time of Sir William Herschel whose "star gauges" led to our first knowledge of the flattened, watch-shaped form of the galactic system. The extension of the gauges to the southern hemisphere by Sir John Herschel in 1834–38 only emphasized their value. The investigations thus begun by the Herschels were continued by Seeliger, Pickering, Chapman and Melotte, and others but especially by Kapteyn,[1] and, more recently, by van Rhijn.[2] Successive revisions of the data have involved not only extensions to larger areas of the sky and to lower limits of brightness, but also improvements in the scale of magnitudes to which the counts are referred.

It is in the scale of magnitudes that the real difficulty lies. To be useful the counts must be to precisely determined limits of brightness. The problem is therefore essentially one of photometry, and, moreover, one in which the conditions are unusually exacting. The range in apparent brightness covered by large telescopes is 20 magnitudes or more—of the same order as the range in intrinsic brightness mentioned above. The establishment of standards of magnitude over this interval therefore requires the ultimate comparison of light sources whose intensities are in the ratio of at least 100,000,000 to 1. More comprehensibly expressed, these numbers are to each other approximately as 1500 miles to 1 inch. Measures of length, however, can be made to a high degree of precision, whereas measures of light, or of energy generally, are among the most difficult and unreliable in the whole field of metrology. Under the most favorable conditions, light sources of the same order of brightness and of the same color can be compared with an uncertainty of about one per cent; but as the differences in brightness and spectral composition increase, systematic errors of various kinds introduce a very much larger uncertainty.

The difficulty of bridging a long interval of brightness by visual methods of measurement is notorious. Thus the visual magnitude scale underlying Kapteyn's table of stellar distribution in *Groningen Publication* No.18—the best scale available up to 1915—is now known to be a magnitude in error at the sixteenth magnitude, the equivalent of 150 per cent in the light-intensity of the stars to which this magnitude was assigned. And thus it happened that the increase in concentration of the stars in the region of the Milky Way with increasing magnitude—a striking feature of stellar distribution already indicated by the gauges of Sir John Herschel[3]—remained an open question until 1917 when counts of 40,000 stars observed photographically at Mount Wilson[4] finally put the matter beyond doubt.

The introduction of photographic methods of photometry, originally through the efforts of Pickering and Miss Leavitt, has greatly increased the precision. The Harvard photographic scale[5] defined by the magnitudes of a sequence of stars at the North Pole is the basis of the distribution table of van Rhijn,[6] hitherto the best available. Investigations begun at Mount Wilson in 1910 soon revealed, however, appreciable errors in the Harvard scale. Continued measures led finally to the photographic Polar standards of brightness published by Seares in 1915.[7] These were subsequently strengthened by the inclusion of new data,[8] and to the sixteenth magnitude the results have been confirmed by measures made at several other observatories; from the sixteenth to the twentieth magnitude, however, the Mount Wilson values are still the only ones available.

The whole investigation, which is the Mount Wilson contribution to the International Scale[9] adopted at Rome in 1922, was preparatory to a study of the distribution of faint stars to be measured in the Selected Areas of

Kapteyn as representative sample regions for the whole sky. This latter undertaking, which has yielded magnitudes for more than 70,000 stars in the 139 areas between the North Pole and declination —15°, is now finished. The magnitudes for individual stars will be published in the *Mount Wilson Catalogue of Selected Areas;* the results on the distribution of stars according to brightness and galactic latitude are incorporated in the table accompanying this communication. The detailed discussion of the counts, which includes a complete revision of the earlier results and an extension of the data to the twenty-first photographic magnitude, will be published in a *Contribution* soon to appear. Only the main features and certain general conclusions need therefore be referred to here.

The new distribution table gives the logarithm of the average number of stars per square degree brighter than magnitude m ($\log N_m$) situated in different galactic latitudes. Local irregularities in density, differences between the northern and southern hemispheres, and systematic deviations in longitude, all of which are still to be investigated, have been ignored. The table depends on: (*a*) The Mount Wilson *Catalogue* referred to above, which completely determines the distribution for stars fainter than $m = 13.5$. (*b*) Van Rhijn's tables, II and IV of *Groningen Publication* No. 27, reduced to the international scale by (*a*). The corrected tables determine the distribution for stars brighter than $m = 9.0$ and the magnitude scale for counts from a third source, (*c*), consisting of 33 zones of the *Astrographic Catalogue* published by Turner.[10] These include about 1,400,000 stars between declinations —65° and +62° and determine the distribution in the interval $m = 9.0$ to 13.5.

The Mount Wilson *Catalogue* (*a*) is based on two partially independent investigations: (a_1) Measures made at Mount Wilson on photographs of 15 minutes exposure or less, which establish the scale and give the magnitudes of 65,683 stars in fields 23′ in diameter; and (a_2) measures at Groningen on Mount Wilson photographs of 60 minutes exposure. These give the magnitudes of 44,910 stars in fields 15′ x 15′ or 20′ x 20′, based on standards derived from (a_1). Most of these stars are included in (a_1), so that the total number of individual objects is approximately 70,400.

The numbers of stars in (a_1) and (a_2) were counted separately for half-magnitude intervals. The counts for both series are either complete, or can safely be completed to a uniform limit of $m = 18.5$. The completed counts were arranged in order of galactic latitude and divided into 28 groups, usually five areas to the group. The sums were formed for each group; then the cumulative sums (N_m) to the successive magnitude limits; and, finally, the logarithms of the cumulative sums, reduced to an area of one square degree.

The results from (a_1) and (a_2) are practically identical. There is no divergence either in latitude or in magnitude. The mean difference in

LOGARITHM OF NUMBER OF STARS PER SQUARE DEGREE TO MAGNITUDE m (LOG N_m) IN DIFFERENT GALACTIC LATITUDES

INT. MAG.	STARS GROUPED ACCORDING TO PHOTOGRAPHIC MAGNITUDE														AVERAGES—PHOT. MAG.				AVERAGE VISUAL
	0°	5°	10°	15°	20°	25°	30°	35°	40°	50°	60°	70°	80°	90°	0–20°	20–40°	40–90°	0–90°	0–90°
4.0	8.19	8.17	8.12	8.05	7.99	7.93	7.87	7.82	7.78	7.74	7.71	7.69	7.67	7.66	8.12	7.88	7.73	7.94	8.11
4.5	8.42	8.40	8.35	8.28	8.22	8.16	8.10	8.05	8.01	7.97	7.94	7.92	7.90	7.88	8.35	8.11	7.96	8.17	8.35
5.0	8.65	8.63	8.58	8.51	8.45	8.39	8.33	8.28	8.24	8.20	8.17	8.15	8.13	8.12	8.58	8.34	8.19	8.40	8.60
5.5	8.88	8.86	8.80	8.74	8.68	8.62	8.56	8.51	8.47	8.43	8.40	8.38	8.36	8.34	8.80	8.57	8.41	8.63	8.83
6.0	9.11	9.08	9.03	8.97	8.90	8.84	8.79	8.74	8.70	8.65	8.62	8.60	8.58	8.57	9.03	8.80	8.64	8.85	9.07
6.5	9.33	9.31	9.26	9.19	9.13	9.07	9.01	8.97	8.92	8.88	8.85	8.83	8.80	8.79	9.26	9.03	8.86	9.08	9.31
7.0	9.56	9.53	9.48	9.42	9.35	9.29	9.24	9.19	9.15	9.10	9.07	9.05	9.02	9.01	9.48	9.25	9.09	9.30	9.54
7.5	9.78	9.76	9.70	9.64	9.57	9.52	9.46	9.41	9.37	9.32	9.29	9.27	9.24	9.23	9.70	9.47	9.31	9.52	9.77
8.0	0.00	9.98	9.92	9.86	9.79	9.74	9.68	9.64	9.59	9.54	9.51	9.48	9.46	9.44	9.92	9.69	9.53	9.74	0.00
8.5	0.23	0.20	0.14	0.08	0.01	9.95	9.90	9.85	9.81	9.76	9.73	9.69	9.67	9.65	0.14	9.91	9.74	9.96	0.23
9.0	0.45	0.42	0.36	0.29	0.22	0.17	0.12	0.07	0.03	9.98	9.94	9.90	9.88	9.86	0.36	0.13	9.56	0.18	0.45
9.5	0.67	0.64	0.57	0.50	0.44	0.38	0.33	0.28	0.24	0.19	0.15	0.11	0.08	0.06	0.58	0.34	0.16	0.39	0.68
10.0	0.89	0.85	0.79	0.72	0.65	0.59	0.54	0.50	0.45	0.40	0.35	0.30	0.28	0.26	0.79	0.55	0.37	0.60	0.90
10.5	1.10	1.07	1.00	0.93	0.86	0.80	0.75	0.70	0.66	0.60	0.55	0.50	0.47	0.45	1.00	0.76	0.57	0.81	1.11
11.0	1.32	1.28	1.21	1.14	1.06	1.01	0.96	0.91	0.86	0.80	0.74	0.69	0.65	0.64	1.22	0.96	0.76	1.02	1.32
11.5	1.53	1.49	1.42	1.34	1.27	1.21	1.16	1.11	1.06	0.99	0.92	0.87	0.84	0.82	1.43	1.17	0.95	1.22	1.53
12.0	1.74	1.70	1.63	1.54	1.47	1.41	1.36	1.30	1.25	1.18	1.11	1.05	1.01	1.00	1.63	1.36	1.14	1.42	1.74
12.5	1.96	1.91	1.83	1.75	1.67	1.61	1.55	1.49	1.44	1.36	1.28	1.23	1.18	1.17	1.84	1.56	1.32	1.62	1.94
13.0	2.16	2.12	2.04	1.95	1.87	1.80	1.74	1.68	1.62	1.54	1.46	1.39	1.35	1.33	2.04	1.75	1.50	1.82	2.14
13.5	2.37	2.32	2.24	2.14	2.06	1.99	1.92	1.86	1.80	1.71	1.62	1.56	1.51	1.49	2.24	1.93	1.67	2.01	2.34
14.0	2.57	2.52	2.43	2.34	2.24	2.17	2.10	2.03	1.97	1.88	1.78	1.72	1.67	1.65	2.44	2.11	1.83	2.20	2.52
14.5	2.77	2.72	2.63	2.52	2.43	2.34	2.27	2.20	2.14	2.04	1.94	1.87	1.82	1.80	2.63	2.29	1.99	2.38	2.71
15.0	2.96	2.91	2.82	2.71	2.60	2.51	2.44	2.36	2.30	2.19	2.09	2.01	1.96	1.94	2.82	2.45	2.14	2.56	2.89
15.5	3.15	3.10	3.01	2.89	2.77	2.68	2.60	2.52	2.45	2.34	2.24	2.15	2.10	2.08	3.01	2.62	2.29	2.73	3.07
16.0	3.33	3.28	3.19	3.07	2.94	2.84	2.75	2.67	2.60	2.48	2.37	2.29	2.23	2.21	3.19	2.77	2.43	2.90	3.24
16.5	3.51	3.46	3.37	3.24	3.10	2.99	2.90	2.81	2.74	2.61	2.50	2.42	2.36	2.34	3.37	2.92	2.56	3.07	3.40
17.0	3.68	3.64	3.54	3.41	3.26	3.14	3.04	2.95	2.87	2.74	2.63	2.54	2.48	2.46	3.54	3.07	2.69	3.23	3.56
17.5	3.85	3.81	3.71	3.57	3.41	3.28	3.17	3.08	3.00	2.86	2.75	2.66	2.60	2.57	3.70	3.20	2.81	3.39	3.71
18.0	4.01	3.97	3.87	3.73	3.56	3.42	3.30	3.20	3.12	2.98	2.86	2.77	2.71	2.68	3.86	3.34	2.93	3.54	3.86
18.5	4.16	4.12	4.03	3.88	3.70	3.55	3.42	3.32	3.23	3.08	2.97	2.88	2.82	2.79	4.02	3.46	3.04	3.68	4.00
19.0	4.32	4.28	4.18	4.02	3.84	3.67	3.54	3.43	3.34	3.19	3.08	2.98	2.92	2.89	4.17	3.59	3.14	3.82	4.13
19.5	4.46	4.42	4.32	4.16	3.97	3.79	3.65	3.53	3.44	3.29	3.17	3.07	3.01	2.98	4.31	3.70	3.24	3.96	4.26
20.0	4.60	4.56	4.46	4.29	4.09	3.90	3.75	3.63	3.53	3.38	3.26	3.16	3.10	3.07	4.45	3.81	3.33	4.09	4.38
20.5	4.74	4.69	4.59	4.42	4.21	4.01	3.85	3.72	3.62	3.46	3.34	3.25	3.18	3.15	4.58	3.91	3.42	4.21	
21.0	4.87	4.82	4.72	4.54	4.33	4.11	3.94	3.81	3.70	3.54	3.42	3.33	3.26	3.22	4.71	4.01	3.50	4.33	

log N_m is $(a_1)-(a_2) = -0.007$, an entirely satisfactory agreement, since the factors used to reduce the counts to unit area are 9 or larger.

The direct means of corresponding values of log N_m from (a_1) and (a_2) were adopted as the "observed" densities for the mean latitudes of the groups. These are affected by a small systematic error arising from the fact that each area is centered on a bright star, and to a very considerable degree by local fluctuations in density from area to area within the same group. The elimination of the error and a graphical process of smoothing and interpolation led finally to the tabulated values of log N_m between the limits $m = 13.5$ and 18.5.

Since the total area of the 139 fields is only 16 square degrees as against more than 40,000 for the entire sky, there is question as to the sufficiency of the samples used to represent the distribution of the stars as a whole. Examination shows that had the 28 latitude groups been divided into two series, one including groups 1, 3, 5, ..., the other, groups 2, 4, 6, ..., the separate discussion of the two series would have led to practically the same result. Individual groups sometimes show large deviations from the smoothed values, but there is little systematic uncertainty. This agreement, perhaps as well as anything, illustrates the underlying structural unity of the stellar system. A study of 1/5000 of the stars accessible to observation, properly selected, will reveal the chief characteristics of distribution, not merely qualitatively, but numerically as well, with only a small percentage uncertainty.

Although van Rhijn's distribution table is based on the Harvard scale, its reduction to the international scale is complicated by the fact that the densities for stars fainter than the tenth magnitude depend on the *Harvard-Groningen Durchmusterung of the Selected Areas*,[11] which is affected by a large color equation. A special comparison of magnitudes from this source with magnitudes for the same stars from the Mount Wilson *Catalogue* led to a mean reduction curve.[12] Combined with results from an investigation of the scale of the bright stars,[13] this gave the following corrections to the magnitude limits of van Rhijn's table:

MAG.	CORR.	MAG.	CORR.	MAG.	CORR.	MAG.	CORR.
4.0	−0.08	8.0	+0.20	12.0	+0.40	16.0	+0.16
5.0	−0.07	9.0	+0.30	13.0	+0.38		
6.0	0.00	10.0	+0.38	14.0	+0.33		
7.0	+0.10	11.0	+0.40	15.0	+0.27		

After the application of these corrections and some smoothing, the interpolation of values of log N_m for half-magnitude intervals gave the tabulated results for the stars brighter than $m = 9.0$. The discussion of the Astrographic Zones, which fill in the gap between magnitudes 9.0 and 13.5, followed a method fully described in a previous paper[14] and need not be considered here.

The final combination of data from the three sources required little systematic adjustment—none at all at $m = 9.0$, and only slight changes at 13.5. The increase in log N_m with increasing m is so regular that the results have been extrapolated from $m = 18.5$ to 21.0. The uncertainty affecting this extension is somewhat in doubt. The error involved in the extrapolation itself cannot be large; the real question is the accuracy of the densities for the faintest stars, and this, in turn, depends on the reliability of the magnitude scale in the interval $m = 16.0$ to 18.5. If the magnitudes at the lower of these limits are systematically correct within 0.1 mag, the precision is probably all that can be hoped for at present.

In addition to the values of log N_m at specified distances from the galactic plane, the table also gives, in the four columns preceding the last, averages for the latitude intervals 0–20°, 20–40°, 40–90°, and 0–90°, and, in the last column, averages over the whole sky for a grouping of stars according to *visual* magnitude. This last series of values was obtained from that in the column preceding by correcting the magnitude limits for the influence of color.[15] The complete transformation of the table into one referred to visual limits of brightness cannot be made at present because of lack of detailed values for the variation in the average color of the stars with galactic latitude.

The very rapid increase in the galactic concentration of the stars with increasing magnitude is a conspicuous feature of the distribution, thus confirming again the early result of Sir John Herschel. At the fourth magnitude the ratio of the numbers of stars per square degree at latitudes 0° and 90° is 3.5. At the twenty-first photographic magnitude the totals per square degree in the Milky Way and at the Pole are 73,600 and 1667, respectively, with a ratio of 44. To the same limit the integrated total for the whole sky is 890,000,000. To the twentieth visual magnitude the corresponding total is 1,000,000,000.

It is well known that with increasing distance from the sun the volume density of stars decreases—the stars thin out in passing to regions more and more remote. This circumstance is reflected in the data of the distribution table, and, in fact, as already mentioned, such data are the basis for determining the rate at which the thinning-out takes place. The revision of the density law must come later, but the total number of stars in the system can at once be estimated, roughly at least, as has already been done by Kapteyn and van Rhijn with the aid of earlier counts, by integrating to $m = +\infty$ the function which represents the rate of change of N_m with m. From the twelfth or thirteenth magnitude downward the values of log (dN_m/dm) for all latitudes can be almost exactly represented by a quadratic formula in m. Whether the law holds for all magnitudes fainter than 21 we have no means of knowing. We can only assume that it does, with full recognition of the fact that the extrapolation involved is extreme.

The integrated totals thus derived from the last two columns of the table are 3.7×10^{10} and 3.5×10^{10}, respectively, results sensibly the same, as they should be. The mean densities for the latitude intervals 0–20°, 20–40°, and 40–90°, treated separately and summed, give 3.0×10^{10}, also in agreement within the numerical uncertainty. If the quadratic formula for log (dN_m/dm) does not hold for stars inaccessible to direct observation, these numbers are likely to be decreased rather than increased. Whatever the actual total, it seems clear that the number of stars brighter than the twenty-first photographic magnitude is but a very small percentage of those still beyond the reach of the most powerful telescopes.

The separate totals for the latitude intervals 0–20°, 20–40° and 40–90° emphasize again the importance of the Milky Way as a structural feature of the system. They show that ninety-five per cent of the stars are within 20° of the galactic circle, or, stated otherwise, that regions centered on the poles of the Milky Way comprising two-thirds of the sky include but five per cent of the stars belonging to our system.

[1] *Publ. Astron. Laboratory at Groningen*, No. 18, 1908.

[2] *Ibid.*, No. 27, 1917.

[3] Results of Astronomical Observations at the Cape of Good Hope, *London*, p. 383 (1847).

[4] Seares, *Mt. Wilson Comm.*, Nos. 43, 49; *Proc. Nat. Acad. Sci., Washington*, **3**, 217–222, 558–562 (1917).

[5] Leavitt, *Cambridge, Mass., Ann. Obs. Harvard Coll.*, **71**, No. 3, 1917.

[6] *Publ. Astron. Laboratory at Groningen*, No. 27, Table IV, 1917.

[7] *Mt. Wilson Contr.*, No. 97; *Astrophys. J., Chicago*, **41**, 206–236 (1915).

[8] Seares and Humason, *Mt. Wilson Contr.*, No. 234; *Astrophys. J., Chicago*, **46**, 84–96. (1922), and Seares, *Mt. Wilson Contr.*, No. 235; *Astrophys. J., Chicago*, **46**, 97–118 (1922).

[9] *Trans. Internat. Astron. Union, London*, **1**, 69 (1922).

[10] Numerous articles in *London, Mon. Nat. R. Astr. Soc.*, since 1909.

[11] *Cambridge, Mass., Ann. Obs. Harvard Coll.*, **101** (1918); **102** (1923); **103** (1924). The first volume alone was used by van Rhijn. The oth rs have only recently been published.

[12] Seares, *Mt. Wilson Contr.*, No. 289; *Astrophys. J., Chicago*, **61** (1925).

[13] Seares, *Mt. Wilson Contr.*, No. 288; *Astrophys. J., Chicago*, **61** (1925).

[14] Seares, *Mt. Wilson Contr.*, No. 135; *Astrophys. J., Chicago*, **46**, 117–137 (1917).

[15] Seares, *Mt. Wilson Contr.*, No. 287; *Astrophys. J., Chicago*, **61**, 114-129 (1925).

THE GENERAL DISTRIBUTION OF COSMICAL VELOCITIES

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Our knowledge of the stellar motions has for a long time been limited to the brighter stars. During the last decade, however, radial velocity determinations have been extended to faint stars and nebulae, and it seems now possible to determine the general form of the distribution law of cosmical velocities. The writer has recently finished an analysis of all the known radial velocities of stars and nebulae, the results of which indicate strongly that the general velocity distribution can be expressed in a very simple form.

The radial velocities are in this case superior to the proper motions because they are given in linear measure, and further because we do not have to wait hundreds of years to get a result, as we must in determining the proper motions of extremely distant objects.

The objects used in this study have been grouped on the basis of spectral type, nebulous character, period of light variation, etc. For spectral types B to M the stars have been subdivided on the basis of proper motion (μ) and apparent magnitude (m), and a certain quantity H has been computed, which is intimately connected with the absolute magnitude (M) and the linear tangential motion (T). We have, for T in km./sec.,

$$H = m + 5 \log \mu = M + 5 \log T - 8.378. \qquad (1)$$

By using H as the basis of grouping it was possible to combine in the same groups stars of a certain spectral type and nearly the same intrinsic brightness and velocity dispersion.

Fifty groups in all were studied, covering all known classes of objects. For each group the velocity distribution was described in terms of the constants of the ellipsoidal distribution law.

$$F(xyz)\,dx\,dy\,dz = \frac{N\,dx\,dy\,dz}{(2\pi)^{3/2}\,abc} \exp\left[-\frac{(x-\bar{x})^2}{2a^2} - \frac{(y-\bar{y})^2}{2b^2} - \frac{(z-\bar{z})^2}{2c^2}\right]. \qquad (2)$$

The velocity components are here represented by x, y and z and are projected on the principal axes of the distribution; N is the number of objects in the group; a, b and c denote the dispersions along the principal axes and $\bar{x}$, $\bar{y}$ and $\bar{z}$ represent the velocity components of the centroid of the group. The principal axes of the velocity distribution were assumed to have the same directions for the different groups, an assumption known to be nearly correct for the stars in general.

The principal result of this analysis is that the group-motion varies greatly, from about 12 km./sec. for the Cepheids of long period to about 300 km./sec. for stars of maximum velocity dispersion, globular clusters, and spiral nebulae. The solar motion, which is the motion of the sun relative to the stars, is thus not a constant vector, but is dependent upon the class of object to which its motion is referred.

The general features of this variability in group-motion are illustrated

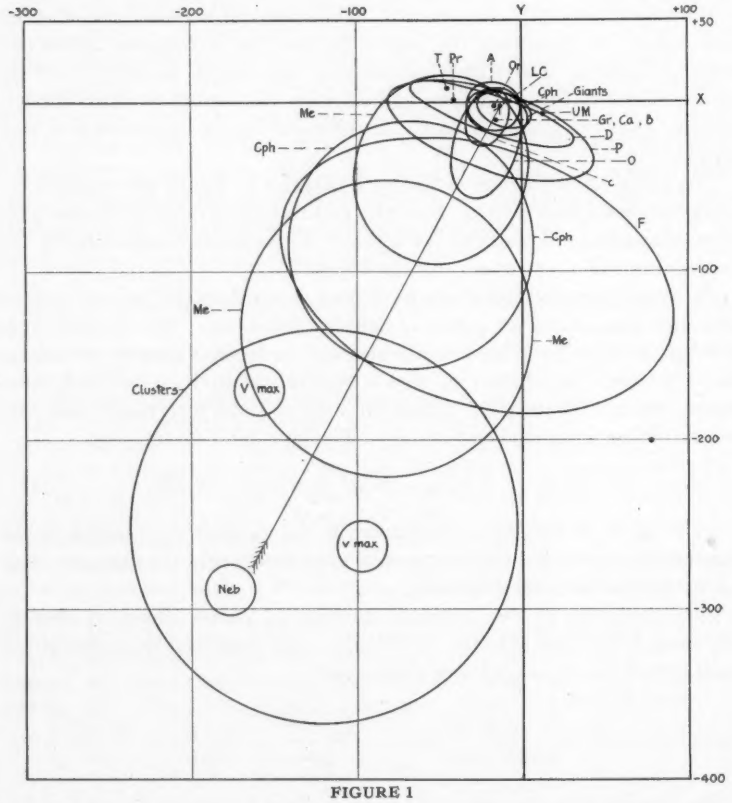

FIGURE 1

in figure 1, which shows the projection of the different velocity ellipsoids in the plane of the galaxy, with the x-axis towards the intersection of the galactic plane with the equator (in Aquila). The sun is at the origin and a vector from the origin to the center of an ellipse or circle indicates the group-motion of a particular class of objects relative to the sun and projected in the galactic plane. The axes of the ellipses indicate the maximum

and minimum values of the internal velocity dispersion. The points represent groups of stars whose motions are nearly parallel and equal to one another (Moving Groups). The small circles marked "V max," "v max" and "Neb" indicate group-motion, but the radii of the circles

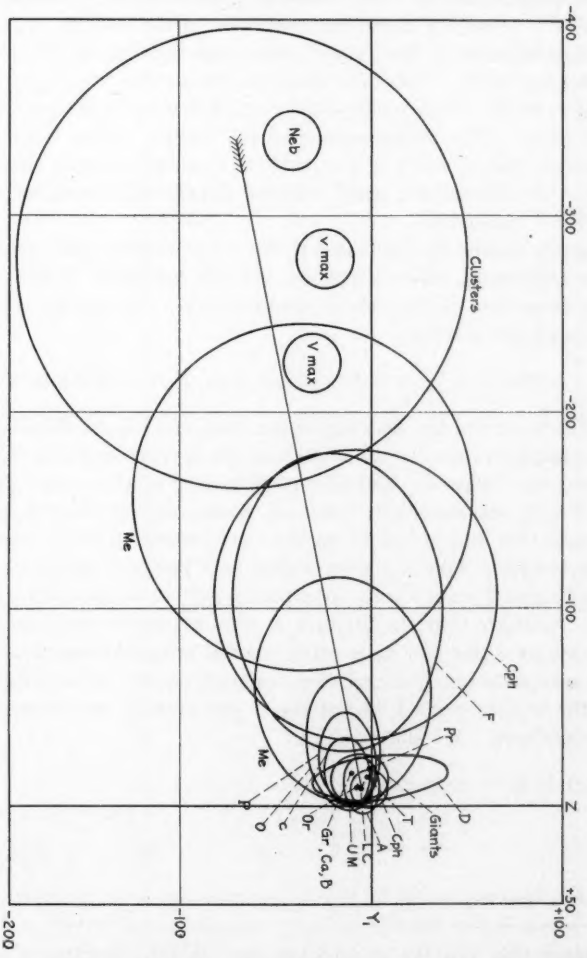

FIGURE 2

have nothing to do with the velocity-dispersion. The first two of these groups are selected directly on the basis of motion and the dispersion is thus affected by the selection; the third group consists of the non-galactic nebulae, the majority of which have spiral structure.

The diagram shows a steady increase in group-motion along a definite axis with increase in dispersion, in harmony with previous results based mainly upon space-velocities.[1] This systematic change in group-motion causes an asymmetry in the distribution of velocities which is of a surprisingly general nature. The direction of this change is in galactic longitude 241.5°. Figure 2 shows the projection of the velocity ellipsoids in a plane perpendicular to the galactic plane and through an axis parallel to the systematic shift. This new projection shows that the displacement of the centers of the velocity ellipsoids is nearly but not quite parallel to the galactic plane. The displacements of the centers define a fundamental axis the direction of which is $\alpha = 310°$, $\delta = +57°$, which is the direction of motion of objects with small velocity dispersion relative to objects of high velocity dispersion.

The group motion does not seem to depend primarily upon spectral type, intrinsic brightness, mass or density, but can be shown to vary with the velocity dispersion of the objects studied. This dependence is expressed by the quadratic relation

$$y = -pb_2 + \beta, \qquad p = 0.0192 \text{ sec./km.}, \beta = -10.0 \text{ km./sec.} \quad (3)$$

The y-axis is here in the direction of the shift, and b is the dispersion along an axis parallel to the y-axis. The relation is represented by a parabola in figure 3, the values of $\bar{y}$ and b being plotted for all the groups studied.

The B stars and the highly luminous A stars do not lie on the parabola, which indicates that so far as motions are concerned these stars form a separate system. This is not surprising, as it has been found by Shapley[2] that the bright B stars form a subordinate system within a larger system, and it is probable that the brighter A stars belong to this system, which thus moves as a unit and has a small internal velocity dispersion.

If we accept the quadratic relation, equation (3), the velocity distribution for all the objects studied, except the B star system, can be written in a very simple form. We find

$$F(xyz)dx\,dy\,dz = dx\,dy\,dz \exp\left[-p(y-\beta)\right] \times \sum_v A_v \exp\left[-\frac{(x-\alpha)^2}{2a_v^2} - \frac{(y-\beta)^2}{2b_v^2} - \frac{(z-\gamma)^2}{2c_v^2}\right]. \quad (4)$$

The function expressed by the summation sign is a symmetrical distribution whose center has the velocity components α, β and γ. In polar coördinates this velocity is 14.8 km./sec. in the direction $\alpha = 85.7°$, $\delta = -22.0°$.

The symmetry of the velocity distribution is now restored, but we have instead introduced the functional factor $\exp\left[-p(y-\beta)\right]$, which seems to be constant for all the groups. If we assume that there is a limit to the

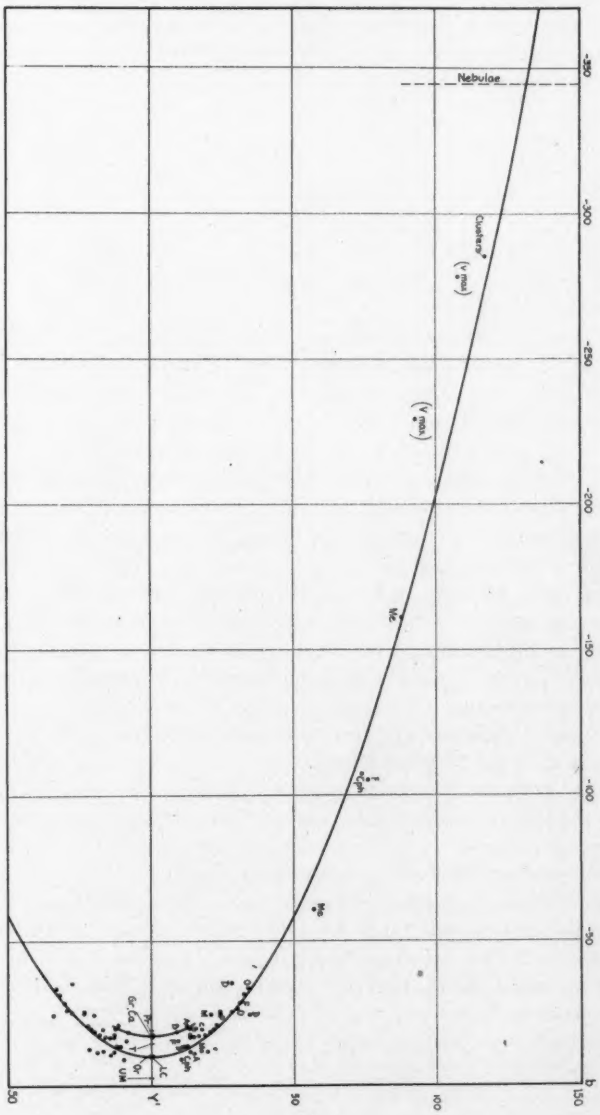

FIGURE 3

series of velocity ellipsoids, and that we have reached this limit when we study the fastest moving stars, the globular clusters, and the spiral nebulae, all of which seem to have the same group-motion, the general velocity distribution can be expressed in the form of a product of two symmetrical distributions, thus

$$F(xyz)dx\,dy\,dz = S_1\,S_2\,dx\,dy\,dz$$

$$S_1 = \sum_v B_v \exp\left[-\frac{(x-\alpha)^2}{2}\left(\frac{1}{a_v^2}-\frac{1}{l^2}\right)-\frac{(y-\beta)^2}{2}\left(\frac{1}{b_v^2}-\frac{1}{l^2}\right)-\frac{(z-\gamma)^2}{2}\left(\frac{1}{c_v^2}-\frac{1}{l^2}\right)\right]$$

$$S_2 = \exp\left[-\frac{(x-\alpha)^2}{2l^2}-\frac{(y-\beta+pl^2)^2}{2l^2}-\frac{(z-\gamma)^2}{2l^2}\right]. \qquad (5)$$

The sun's velocity relative to the center of the distributions S_1 is 14.8 km./sec. in the direction $\alpha = 265.7°$, $\delta = +22.0°$. Its velocity relative to the center of the distribution S_2, which is identical with its velocity relative to fast moving stars, clusters and spirals, is found to be about 300 km./sec. in the direction $\alpha = 307°$, $\delta = +56°$. With this velocity known, we can determine the value of l, which is the dispersion in the distribution S_2. We thus find $l = 123$ km./sec., which is about the same as the dispersion for the globular clusters (117 km./sec.).

The physical meaning of the general distribution function is that the motions of the stars in our neighborhood are determined not only by the gravitational field due to a local system of stars, but also by another field connected with the system of globular clusters and spiral nebulae. The function S_2 may be due to the general gravitational field of this vast system. There are many difficulties in the way of this explanation, and if this second field is found incapable of producing such an asymmetrical velocity distribution, the only alternative seems to be the assumption of a fundamental reference frame or medium referred to which high cosmic velocities are non-existant. We must in this connection remember that a star is not a simple object, but is itself a complicated system of atomic nuclei and electrons, which already have extremely high relative velocities. Further, the observed motion of a star is due to its initial velocity and the accumulated effect of all the impulses during its whole life time. We are hardly justified in saying that the nature of all these impulses, even the most feeble ones, are known to us.

[1] *Astrophys. J., Chicago*, **59**, 228 (1924). These PROCEEDINGS, **9**, 312 (1923).

[2] *Astr. Nachr., Kiel*, Jubiläumsnummer, p. 25 (1921).